Open questions about K3 surfaces & friends
Lorentz Center, Leiden
July 2025

July 13, 2025

1 K3 surfaces

1. (E. Markman) Hodge conjecture for a product of two K3 surfaces. The case of Hodge isometries is known (Buskin). What about Hodge similitudes?

2. (A. Sarti, Gizatullin's question) If X is a quatric K3 surface of Picard rank $\rho \geq 3$, when are automorphisms of X induced by Cremona transformations? (Work by Oguiso, and more recently by by Araujo–Paiva–Zikas in Picard rank 2)

3. (D. Holmes) What is a log K3 surface? Is it possible to give modular interpretation of all toroidal compactifications of moduli of polarised K3 in terms of some kind of log K3? This is motivated by the case of abelian varieties, where all toroidal compactifications are given log modular interpretations by work of Kajiwara, Kato, and Nakayama, see Kajiwara–Kato—Nakayama, and by the tropical moduli of K3, see Odaka– Oshima.

4. (E. Shinder) What is the largest degree of a primitive polarization on a K3 surface defined over $\mathbb{Q}$ of geometric Picard rank 1? These should be bounded by the Shafarevich conjecture.

5. (G. Mezzedimi) Can we write down examples of K3s over $\mathbb{Q}$ with geometric Picard rank 1 and degree ≥ 6? There are examples in degree 4 by van Luijk and degree 2 by Elsenhans–Jahnel.

6. (F. Gounelas) Given X K3 over a number field, is there always a prime p of good reduction where the reduction is elliptic and not supersingular?

7. (R. Meinsma, E. Shinder) Relationship between L-equivalence and D-equivalence for K3 surfaces? L-equivalence implied D-equivalence for very general K3s by the work of Meinsma. What about the other implication, which is known to fail for higher-dimensional HK manifolds.

2 Rational points and Brauer–Manin obstruction

1. (D. Festi) In the literature there are several examples of varieties over global fields, for which the failure of the local–global principle is partially explained by the (étale) Brauer–Manin obstruction (e.g. by work of Skorobogatov, Poonen, Colliot-Thélène–Pál–Skorobogatov, Harpaz–Skorobogatov, Kebekus–Pereira–Smeets). We have some explicit equations for these varieties; can we write down more explicit examples?

2. (J. Petok, based on Debarre–Laface–Roulleau) Y smooth cubic 4-fold over $\mathbb{F}_3$. Does Y have a line over $\mathbb{F}_3$? The question is asked in the paper of Debarre–Laface–Roulleau, where they prove this for any other finite field.

3. (N. Addington) Let Y be an intersection of two cubics in $\mathbb{P}^5$ containing an abelian surface A as described in my paper with D. Bragg. Then $X = \mathrm{Bl}_A(Y)$ is a CY 3-fold fibered in abelian surfaces, with $\mathrm{Br}(X) \supseteq (\mathbb{Z}/3\mathbb{Z})^2$. Can we produce Brauer obstructions for rational points or weak approximation? Gross and Pavanelli showed that a similar example involving four quadrics in $\mathbb{P}^7$ has $\mathrm{Br}(X) = (\mathbb{Z}/8\mathbb{Z})^2$. More examples along the same lines appear in an earlier paper of Gross and Popsecu, but you would have to analyze the Brauer groups yourself.

4. (A. Skorobogatov) k global field of char $p \geq 3$, e.g. $k = \mathbb{F}_p(t)$. E supersingular elliptic curve over k. $X = \mathrm{Kum}(E \times E)$. Can we write down explicitly the Brauer–Manin set in terms of an Azumaya algebra?

5. (A. Skorobogatov) An old conjecture is that Brauer–Manin obstructions control rational points on K3 surfaces over number fields. Is there a better chance to check this over global fields of positive characteristic as we have more geometric tools in that case?

3 Derived categories

1. (A. Kuznetsov) A K3 category is a smooth and proper category $\mathcal{A}$ such that the Serre functor $S_{\mathcal{A}}$ is [2], and which is deformation equivalent to a derived category of a K3 surface (alternatively one can ask for the Hochschild homology of $\mathcal{A}$ to be the same as for a K3 surface). Currently there are three examples of K3 categories coming from (1) cubic fourfolds, (2) Gushel–Mukai fourfolds and sixfolds, (3) Debarre–Voisin 20-folds. In all these examples there is a 20-dimensional family of K3 categories. Can we construct more examples? What about HK categories? Noncommutative abelian varieties? Calabi–Yau categories of negative dimension?

2. (N. Addington) The derived category of a complex Enriques surface S has no full exceptional collection, because $K(S) = \mathbb{Z}^{12} \oplus \mathbb{Z}/2$. But if α is the non-trivial element of $\mathrm{Br}(S) = \mathbb{Z}/2$, then $K(S, \alpha) = \mathbb{Z}^{12}$, so is there a full exceptional collection in $D(X, \alpha)$? Or a phantom category? By work of Ingalls and Kuznetsov and of Hosono and Takagi, the derived category of an Artin–Mumford quartic double solid X admits a semiorthogonal decomposition containing $D(S)$. If β is the non-trivial element of $\mathrm{Br}(X) = \mathbb{Z}/2$, is there a similar relation between $D(X, \beta)$ and $D(X, \alpha)$? Note that $D(X, \beta)$ has a full exceptional collection, as discussed in my paper with E. Elmanto, §3.2.

3. (N. Addington) The Brauer group of an Enriques surface over $\mathbb{C}$ is $\mathbb{Z}/2$. Can we write down an explicit $\mathbb{P}^1$-fibration representing the non-trivial element? For an explicit description the surface itself, see Beauville's book on surfaces, Example VIII.18: let $\mathbb{Z}/2$ act on $\mathbb{P}^5$ diagonally with weights $(1, 1, 1, -1, -1, -1)$, and take a general intersection of three invariant quadrics to get a K3 surface with a free involution, and take the quotient.

4. (D. Mattei) X, Y HK of $K3^{[n]}$-type. Is it true that the derived categories of X and Y are equivalent iff there is a Hodge isometry $\Lambda_X \cong \Lambda_Y$ (where Λ_X is $U \oplus H^2(X, \mathbb{Z})$ as a lattice, with a twisted Hodge structure) There is work by Taelman and Beckmann, where they prove one direction.

4 HK manifolds

1. (E. Markman) Classification of HK varieties. Can we either show that no new classes exist by proving that every HK manifold is deformation equivalent to a quotient of abelian variety (Laza–Kim–Martin)? On the other hand, can construct new examples as moduli spaces of atomic sheaves on HK manifolds? (Example of OG10 by Bottini.)

2. (D. Huybrechts) How many twistor lines does it take to connect two K3 surfaces on the 20-dimensional moduli of K3 surfaces. Is this a finite number? For two K3s "close enough" the number should be 3.

3. (D. Huybrechts) It is known by a result of Huybrechts that starting with a projective K3 with CM, on its twistor sphere all algebraic fibres away from the equator are CM, and the corresponding CM endomorphism fields share the same totally real maximal subextension. In Vigano's paper, the result was extended to show that along the latitude, the CM field remains constant. What is the geometric reason behind these observations? What are the CM fields that can show up? What are the distances between the latitude lines where the algebraic/CM K3s lie?

4. (D. Huybrechts) Is there a geometric interpretation of the algebraic points on the twistor sphere in terms of the original K3? This is known to exist for the degenerate twistor (or the Tate–Shafarevich) line.

5. (F. Gounelas, repeating a question by C. Voisin) X smooth projective HK. Can X be covered by curves of (geometric) genus 1? Known for K3s, Hilbert schemes of points on K3s and generalized Kummers. E.g. what about $\mathrm{K3}^{[2]}$-type, specifically, Fano varieties of lines on cubic 4-folds?

6. (Y. Dutta) If S is a quartic K3 containing a line, then the projection from the line gives an elliptic fibration. Now let X be a quartic 8-fold containing a $\mathbb{P}^3$. The projection gives a family $\mathrm{Bl}_{\mathbb{P}^3}(X) \to \mathbb{P}^5$ of cubic threefolds. Take the relative intermediate Jacobian $J_U \to U$ over an open subset $U \subseteq \mathbb{P}^5$. Is there any 10-dimensional HK lying around? Same question if X is a quartic in $\mathbb{P}^{27}$ containing a $\mathbb{P}^5$.

7. (Y. Dutta) Let X be a quartic 8-fold containing a $\mathbb{P}^4$. The projection gives a family $\mathrm{Bl}_{\mathbb{P}^4}(X) \to \mathbb{P}^4$ of cubic fourfolds. Take the relative LSV

construction $M_U \to U$ over an open subset $U \subseteq \mathbb{P}^4$. This is a family of OG10. Can this be completed? What are the degenerate OG10s appearing as limits in this family?

8. (D. Huybrechts) Is the cohomology ring of the known HK manifolds torsion free? What about the H^3? Known for $\mathrm{K3}^{[n]}$ by work of Markman and Kum_2 by work of Kapfer–Menet.

9. (D. Gvirtz-Chen) Let X be a normal K3 over $\mathbb{C}$ (so its minimal resolution is K3). The triviality of K_X implies that all singularities of X are canonical. A theorem due to Campana, first conjectured by D.-Q. Zhang, states that the universal cover of X^{sm} is either another K3 (and $\pi_1(X^{sm})$ is finite) or it factors through a big open subset of a torus. Is the statement true if one replaces K3 with HK? An earlier paper by Catanese–Keum–Oguiso proves the result for elliptic K3 surface X. Can the result be proven for HK with a Lagrangian fibration?

10. (F. Gounelas) $D \subseteq X$ over $\mathbb{C}$ an snc divisor, where X is K3 or HK. How positive is $\Omega_X(\log D)$?

11. (D. Holmes) What is the log-fundamental group of a log-HK? For this you can decide the definition of log HK, but probably the most interesting case is that of a vertical log structure.