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Summer term 2023

Exercises, Algebraic Geometry II – Week 1

Exercise 1. *Fundamental Lemma of Homological Algebra* (4 points)
Let $\mathcal{C}$ be an Abelian category. Consider the commutative diagram of solid arrows

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \xrightarrow{d_A^0} & I^0 & \xrightarrow{d_A^1} & I^1 & \xrightarrow{d_A^2} & I^2 & \xrightarrow{d_A^3} & \dots \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \\ 0 & \longrightarrow & B & \xrightarrow{d_B^0} & J^0 & \xrightarrow{d_B^1} & J^1 & \xrightarrow{d_B^2} & J^2 & \xrightarrow{d_B^3} & \dots, \end{array}$$

where the first row is exact, the second row is an injective resolution of B, and $f : A \to B$ is a morphism in $\mathcal{C}$. Show that the following hold:

(1) There exist dotted arrows f^i making the diagram commute.

(2) If f'^i is another collection of arrows making the diagram commute, then $(f^i)_{i=0}^{\infty}$ and $(f'^i)_{i=0}^{\infty}$ are *chain homotopic*, i.e., there exists a collection of morphisms $h^i : I^i \to J^{i-1}$ such that
$$f^i - f'^i = d_B^i \circ h^i + h^{i+1} \circ d_A^{i+1}.$$

Exercise 2. *Injective resolutions of groups and modules* (5 points)
Let A be a ring and I an A-module.

(1) Show that I is injective if for any ideal $\mathfrak{a} \subseteq A$, the induced map $\mathrm{Hom}_A(A, I) \to \mathrm{Hom}_A(\mathfrak{a}, I)$ is surjective.

(2) Show that any *divisible* Abelian group G (i.e. such that the map $g \mapsto ng$ is surjective for all $n > 0$) is an injective object in Ab. Deduce that, in particular, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

(3) Show that $I(G) := \prod_{J(G)} \mathbb{Q}/\mathbb{Z}$ is a divisible group, where $J(G) := \mathrm{Hom}_{\mathrm{Ab}}(G, \mathbb{Q}/\mathbb{Z})$.

(4) Show that the natural map $G \to I(G), g \mapsto (f(g))_{f \in J(G)}$ is injective. Conclude that Ab has enough injectives.

(5) Prove that Mod_A has enough injectives for every ring A.

Due 14.04.2022, 2pm

Exercise 3. *Injective objects* (3 points)
Let $\mathcal{C}$ and $\mathcal{D}$ be Abelian categories.

(1) Show that any product of injective objects in $\mathcal{C}$ is again injective.

(2) Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be additive functors such that F is exact and G is right-adjoint to F. Show that G sends injective objects to injective objects.

Exercise 4. *Flasque sheaves* (4 points)
A sheaf of Abelian groups $\mathcal{F}$ on a topological space X is called *flasque* if for every inclusion of open sets $V \subseteq U \subseteq X$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. Show the following:

(1) A constant sheaf on an irreducible topological space is flasque.

(2) If
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
is an exact sequence of sheaves and $\mathcal{F}'$ is flasque, then for every open $U \subseteq X$, the induced sequence
$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$$
is exact.

(Hint: Use Zorn's lemma.)

(3) If
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
is an exact sequence of sheaves and if $\mathcal{F}'$ and $\mathcal{F}$ are flasque, then so is $\mathcal{F}''$.

(4) If $f : X \to Y$ is a continuous map and $\mathcal{F}$ is a flasque sheaf on X, then $f_*\mathcal{F}$ is flasque on Y.

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 5. *An injective non-flasque quasi-coherent sheaf*
Study TAG 0273 of the Stacksproject. It gives an example of a scheme X and an injective object in $\mathrm{QCoh}(X, \mathcal{O}_X)$ which is not flasque as a sheaf.