Dr. Gebhard Martin
Dr. Yajnaseni Dutta

Summer term 2023

Exercises, Algebraic Geometry II – Week 10

Exercise 46. *Mittag–Leffler* (4 points)
Let $(A_n) := (\{A_i\}_{i\in I}, \{\rho_{ij}\}_{i,j\in I, i\geq j})$ be an inverse system of Abelian groups indexed by $\mathbb{N}$. We say that (A_n) satisfies the *Mittag–Leffler condition* if, for each n, the sequence $\{\rho_{in}(A_i) \subseteq A_n | i \geq n\}$ of A_n becomes stationary for large i. For such an inverse system, define $A'_n := \rho_{in}(A_i)$ for $i \gg 0$.

(1) Define maps making the collection of A'_n into an inverse system indexed by $\mathbb{N}$. Show that $\varprojlim A'_n \cong \varprojlim A_n$.

(2) Show that inverse systems with surjective transition maps satisfy the Mittag–Leffler condition.

(3) Let
$$0 \to (A_n) \to (B_n) \to (C_n) \to 0$$
be a short exact sequence of inverse systems of Abelian groups indexed by $\mathbb{N}$. Show that:

   (a) If (B_n) satisfies the Mittag–Leffler condition, so does (C_n).
   (b) If (A_n) satisfies the Mittag–Leffler condition, then
   $$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$$
   is exact.

Exercise 47. *Proper + quasi-finite = finite* (4 points)
Let $f : X \to Y$ be a morphism of Noetherian schemes. Show that f is finite if and only if it is quasi-finite and proper.

Exercise 48. *Connected components of fibers* (4 points)
Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let $y \in Y$ be a point. Show that the connected components of X_y are in bijection with the maximal ideals in $(f_*\mathcal{O}_X)_y$.

Exercise 49. *Analytic isomorphisms* (4 points)
Let X and Y be varieties over an algebraically closed field k. We say that a closed point $x \in X$ is *analytically isomorphic* to $y \in Y$ if there exists an isomorphism of k-algebras $\mathcal{O}^\wedge_{X,x} \cong \mathcal{O}^\wedge_{Y,y}$.

(1) Show that any two closed points on smooth varieties of the same dimension are analytically isomorphic.

(2) Show that if two closed points on curves are analytically isomorphic, then they have the same δ-invariant (see Exercise 29).

   (You may use that normalization commutes with completion in this setting.)

Due 23.06.2023, 2pm

A closed point $x \in X$ is called *hypersurface singularity*, if there exists $f \in k[[x_1, \ldots, x_n]]$ such that $\mathcal{O}^\wedge_{X,x} \cong k[[x_1, \ldots, x_n]]/(f)$. We can write $f = \sum f_r$ with f_r homogeneous of degree r. The smallest r with $f_r \neq 0$ is called *multiplicity* of $x \in X$.

(3) Show that two analytically isomorphic hypersurface singularities have the same multiplicity.

(4) Let $x \in X$ be a hypersurface singularity of multiplicity r of a curve X (we call such a singularity a *planar curve singularity*). Assume that $\mathcal{O}^\wedge_{X,x} \cong k[[x, y]]/(f)$ with $f = \sum_{i=r}^{\infty} f_i$ such that $f_r = g_s h_t$ for g_s and h_t homogeneous without common factors. Show that there are formal power series $g = \sum_{i=s}^{\infty} g_i, h = \sum_{i=t}^{\infty} h_i \in k[[x, y]]$ such that $f = gh$. Conclude that a planar curve singularity of multiplicity 2 such that f_2 is not a square is analytically isomorphic to $(x, y) \in \operatorname{Spec} k[x, y]/(xy)$.

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 50. *Classification of planar curve singularities of multiplicity* 2 (+ 4 extra points) Let k be an algebraically closed field of characteristic different from 2. Show that, for every planar curve singularity $x \in X$ of multiplicity 2, there exists an $n \geq 2$ such that $x \in X$ is analytically isomorphic to $(x, y) \in \operatorname{Spec} k[x, y]/(y^2 - x^n)$.

(Hint: Look up the Weierstrass preparation theorem for complete Noetherian local rings)