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Exercises, Algebraic Geometry II – Week 11

Exercise 51. *Picard group of projective bundles* (4 points)
Let X be a regular Noetherian scheme and let $\mathcal{E}$ be a locally free sheaf of rank ≥ 2 on X.

(1) Show that $\mathrm{Pic}(\mathbb{P}(\mathcal{E})) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$, where $\mathrm{Pic}(X)$ is identified with its image under the pull-back and $\mathbb{Z}$ is generated by $\mathcal{O}(1)$.

(2) If $\mathcal{E}'$ is another locally free coherent sheaf on X, show that $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}')$ if and only if there exists an invertible sheaf $\mathcal{L}$ on X with $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$.

Exercise 52. *Relative Euler sequence* (4 points)
Let Y be a Noetherian scheme and let $\mathcal{E}$ be a locally free sheaf of rank $n+1$ with $n \geq 1$ on Y. Let $X = \mathbb{P}(\mathcal{E})$ and let $\pi : X \to Y$ be the natural projection.

(1) Show that $R^i\pi_*\mathcal{O}_X(d) = 0$ for $0 < i < n$ and $R^n\pi_*\mathcal{O}_X(d) = 0$ for $d > -n-1$.

(2) Show that there is an exact sequence
$$0 \to \Omega_{X/Y} \to \pi^*\mathcal{E}(-1) \to \mathcal{O}_X \to 0.$$
Conclude that the *relative dualizing sheaf* $\omega_{X/Y} := \bigwedge^n \Omega_{X/Y}$ is isomorphic to the sheaf $(\pi^* \bigwedge^{n+1} \mathcal{E})(-n-1)$ and show that $R^n\pi_*\omega_{X/Y} \cong \mathcal{O}_Y$.

(3) For any $d \in \mathbb{Z}$, show that $R^n\pi_*\mathcal{O}_X(d) \cong \pi_*(\mathcal{O}_X(-d-n-1))^\vee \otimes_{\mathcal{O}_Y} (\bigwedge^{n+1} \mathcal{E})^\vee$.

(4) Conclude that if Y is a smooth projective variety over a field k, then $h^0(X, \omega_{X/k}) = 0$.

Exercise 53. *Picard group of blow-ups* (4 points)
Let X be a smooth variety over a field k and let $Y \subseteq X$ be a smooth integral closed subscheme of codimension $r \geq 2$. Let $\pi : \widetilde{X} \to X$ be the blow-up of X in Y and let $Y' = \pi^{-1}(Y)$.

(1) Show that the maps $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$ and $\mathbb{Z} \to \mathrm{Pic}(\widetilde{X}), n \mapsto \mathcal{O}_{\widetilde{X}}(nY')$ determine an isomorphism $\mathrm{Pic}(\widetilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$.

(2) Show that $\omega_{\widetilde{X}/k} \cong \pi^*\omega_{X/k} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{X}}((r-1)Y')$.
(Hint: Use (1) to write $\omega_{\widetilde{X}/k} \cong \pi^*\omega_{X/k} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{X}}(qY')$ for some q and determine q by restricting to a fiber of π over a closed point of Y)

Exercise 54. *Some explicit blow-ups* (4 points)
Let k be a field of characteristic different from 2. Describe the blow-up of X in Y in the following situations (draw pictures!):

(1) $X = \mathrm{Spec}\, k[x,y]/(y^2 - x^n)$ and $Y = V(x,y)$ for $n \geq 2$.

Due 30.06.2023, 2pm

(2) $X = \operatorname{Spec} k[x,y,z]/(z^n + xy)$ and $Y = V(x,y,z)$ for $n \geq 2$.

(3) $X = \operatorname{Spec} k[x,y,z]/(x^2 + y^2 - z^2)$ and $Y = V(x, y - z)$.

(4) $X = \operatorname{Spec} k[x,y]$ and $Y = V(x, y^2)$.

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 55. $\mathbb{P}^n$-*bundles* (+ 4 extra points)
Let X be a Noetherian scheme. A $\mathbb{P}^n$-*bundle* over X is a scheme P together with a morphism $\pi : P \to X$ such that there exists an open affine cover $X = \cup U_i$ and isomorphisms $\psi_i : P \times_X U_i \cong \mathbb{P}^n_{U_i}$ that identify π_{U_i} with the natural projection $\mathbb{P}^n_{U_i} \to U_i$ and such that, on $U_i \cap U_j$, the automorphisms $\psi_j \circ \psi_i^{-1}$ are given by linear automorphisms of the homogeneous coordinate ring of $\mathbb{P}^n_{U_i \cap U_j}$.

(1) Show that if $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on X, then $\mathbb{P}(\mathcal{E}) \to X$ is a $\mathbb{P}^n$-bundle over X.

(2) Assume that X is regular. Show that every $\mathbb{P}^n$-bundle on X is of the form $\mathbb{P}(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ of rank $n+1$.

(3) Can you find a counterexample to (2) if X is not regular?