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Summer term 2023

Exercises, Algebraic Geometry II – Week 12

Exercise 56. *$\mathbb{P}^n$-fibrations* (4 points)
Let $f : X \to Y$ be a projective flat morphism of Noetherian schemes. Assume that Y is integral and all geometric fibers are isomorphic to projective n-space.

(1) Show that f is the $\mathbb{P}^n$-bundle associated to a locally free sheaf on Y if and only if there exists an invertible sheaf $\mathcal{L}$ on X that restricts to $\mathcal{O}(1)$ on all geometric fibers of f.

(2) Conclude that if $n = 1$ and Y is regular and admits an ample invertible sheaf, then $X \cong \mathbb{P}(\mathcal{E})$ over Y for some locally free sheaf $\mathcal{E}$ of rank 2 if and only if f has a section.

Exercise 57. *Tsen's theorem and C_1-fields* (4 points)
A field k is called C_1 if every homogeneous polynomial $f \in k[x_1, \ldots, x_n]$ of degree $d < n$ has a non-trivial zero in k^n.

(1) Show that every finite field extension of a C_1-field is C_1.

(2) Show that if k is algebraically closed, then $k(x)$ is C_1.

(3) Let $f : X \to Y$ be a projective flat morphism of schemes over an algebraically closed field k. Assume that Y is a smooth projective curve over k and that all geometric fibers of f are isomorphic to $\mathbb{P}^1$. Show that there exists a locally free sheaf $\mathcal{E}$ of rank 2 on Y such that $X \cong \mathbb{P}(\mathcal{E})$ over Y.

Exercise 58. *Hodge bundle* (4 points)
Let $f : X \to Y$ be a proper flat morphism of Noetherian schemes. Assume that the fibers of f are connected smooth curves of genus g. Show that $R^1 f_* \mathcal{O}_X$ is locally free of rank g.

(Using relative duality, one can also show that $R^1 f_* \mathcal{O}_X \cong f_* \Omega_{X/Y}$.)

Exercise 59. *Rigidity* (4 points)
Let k be a field and let X, Y, and Z be varieties over k. Let $f : X \to Y$ and $g : X \to Z$ be proper morphisms such that $f_* \mathcal{O}_X = \mathcal{O}_Y$ and such that g contracts every fiber of f (i.e. such that $g(f^{-1}(y))$ is a point for all $y \in Y$). Show that there exists a morphism $h : Y \to Z$ such that $h \circ f = g$.

(Hint: Consider the image of $(f, g) : X \to Y \times_{\mathrm{Spec}\, k} Z$.)

Due 07.07.2023, 2pm

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 60. *The Jouanolou trick* (+ 4 extra points)
Let k be an algebraically closed field. Consider the set Y of all matrices $A \in M_{n+1}(k)$ of rank one satisfying $A^2 = A$.

(i) Show that Y is the set of closed points of an affine variety over k such that the morphism $\pi\colon Y \to \mathbb{P}^n_k$, $A \mapsto \mathrm{Im}(A)$ is a well-defined morphism of schemes.

(ii) Show that the fibers of π are isomorphic to $\mathbb{A}^n_k$ and that, in fact, π is a vector bundle.

(iii) Conclude that for every quasi-projective variety X over k, there exists a vector bundle $\mathbb{V}(\mathcal{E}) \to X$ such that $\mathbb{V}(\mathcal{E})$ is an affine scheme.