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Exercises, Algebraic Geometry II – Week 3

Exercise 11. *Affine plane without origin revisited* (4 points)
Let k be a field and $Y = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ and let $X = Y \setminus \{(0,0)\}$. Consider the open cover $\mathcal{U} = (D(x), D(y))$ of X. Show that there is an isomorphism of k-vector spaces

$$\check{H}^1_{\mathcal{U}}(X, \mathcal{O}_X) \cong \langle \{x^i y^j \mid i,j < 0\} \rangle_k.$$

(By Serre vanishing and the comparison results between Čech- and sheaf cohomology, this gives a new proof that X is not affine.)

Exercise 12. *Picard group and cohomology of $\mathcal{O}_X^\times$* (4 points)
Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of X. Let $\operatorname{Pic}_{\mathcal{U}}(X) \subseteq \operatorname{Pic}(X)$ be the subgroup of the Picard group of X consisting of isomorphism classes of invertible sheaves $\mathcal{L}$ such that there exist isomorphisms $\varphi_i : \mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$ for all $i \in I$.

(1) Show that the collection of $\varphi_j|_{U_i \cap U_j} \circ \varphi_i^{-1}|_{U_i \cap U_j}$ defines an element of $\check{H}^1_{\mathcal{U}}(X, \mathcal{O}_X^\times)$ which depends only on the isomorphism class of $\mathcal{L}$ (and not on the choice of φ_i).

(2) Show that the induced map $\operatorname{Pic}_{\mathcal{U}}(X) \to \check{H}^1_{\mathcal{U}}(X, \mathcal{O}_X^\times)$ is an isomorphism of groups.

(One can show that $\varinjlim \check{H}^1_{\mathcal{U}}(X, \mathcal{O}_X^\times) \cong H^1(X, \mathcal{O}_X^\times)$, where the direct limit is taken over all open covers, ordered by refinement (see Hartshorne, Exercise III.4.4), which, combined with the results of this exercise, yields an isomorphism $\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$)

Exercise 13. *Class group of A_{n-1}* (4 points)
Let k be a field, $n \geq 1$, $Y = \operatorname{Spec} k[x,y,z]/(z^n - xy)$ and $X = Y \setminus \{(0,0,0)\}$.

(1) Show that
$$\operatorname{Pic}(D(x) \cap Y) = \operatorname{Pic}(D(y) \cap Y) = \operatorname{Pic}(D(xy) \cap Y) = 0.$$

(2) Use Exercise 12 to show that
$$\operatorname{Cl}(Y) \cong \operatorname{Pic}(X) \cong \mathbb{Z}/n\mathbb{Z}.$$

Exercise 14. *A formula of Deligne* (4 points)
Let A be a Noetherian ring, $X = \operatorname{Spec} A$, $\mathfrak{a} \subseteq A$ an ideal, and $U := X \setminus V(\mathfrak{a}) \subseteq X$.

(1) Show that, for any A-module M, there is a natural isomorphism
$$\Gamma(U, \widetilde{M}) \cong \varinjlim_n \operatorname{Hom}_{\mathrm{Mod}_A}(\mathfrak{a}^n, M).$$

(Hint: Show directly or use the identification $\Gamma_{V(\mathfrak{a})}(X, \widetilde{M}) \cong \Gamma_{\mathfrak{a}}(M) \cong \varinjlim_n \operatorname{Hom}_A(A/\mathfrak{a}^n, M)$ where the first isomorphism follows fom AGI Exer. 41 (iv).)

Due 28.04.2023, 2pm

(2) Conclude that if I is an injective A-module, then $\widetilde{I}$ is a flasque sheaf on X.

(3) Conclude that, on Noetherian schemes, sheaf cohomology can be calculated by right-deriving the functor $\Gamma(X, -) : \mathrm{QCoh}(X, \mathcal{O}_X) \to \mathrm{Ab}$.

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 15. *Torsors and* H^1 (+ 4 extra points)
Let X be a topological space and let $\mathcal{G}$ be a sheaf of Abelian groups on X. A $\mathcal{G}$*-torsor* is a sheaf of sets $\mathcal{F}$ on X together with a morphism of sheaves of sets $\rho : \mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(a) For all $U \subseteq X$ open, $\rho(U)$ is an action of $\mathcal{G}(U)$ on $\mathcal{F}(U)$ (i.e. ρ is an action in $\mathrm{Sh}(X)$).

(b) For all $U \subseteq X$ open such that $\mathcal{F}(U) \neq \emptyset$, $\rho(U)$ is simply transitive.

(c) For all $U \subseteq X$ open, there exists an open cover $U = \cup U_i$ such that $(\mathcal{F}|_{U_i}, \rho|_{U_i})$ is a trivial $\mathcal{G}|_{U_i}$-torsor.

Here, a *morphism of* $\mathcal{G}$*-torsors* is a morphism of sheaves that is compatible with the action ρ and a $\mathcal{G}$-torsor is called *trivial* if is isomorphic to the torsor $\mathcal{G}$ endowed with the action of left multiplication: $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$.

(1) Show that every morphism of $\mathcal{G}$-torsors is an isomorphism.

(2) Show that a $\mathcal{G}$-torsor $(\mathcal{F}, \rho)$ is trivial if and only if $\mathcal{F}(X) \neq \emptyset$.

(3) Let $\mathrm{Ext}^1(\underline{\mathbb{Z}}, \mathcal{G})$ be the set of isomorphism classes of extensions of sheaves

$$0 \to \mathcal{G} \to \mathcal{E} \to \underline{\mathbb{Z}} \to 0.$$

Show that the map

$$\begin{aligned} \mathrm{Ext}^1(\underline{\mathbb{Z}}, \mathcal{G}) &\to H^1(X, \mathcal{G}) \\ (0 \to \mathcal{G} \to \mathcal{E} \to \underline{\mathbb{Z}} \to 0) &\mapsto \delta(1), \end{aligned}$$

where $1 \in H^0(X, \underline{\mathbb{Z}})$, is a bijection.

(4) Given an extension

$$0 \to \mathcal{G} \to \mathcal{E} \xrightarrow{\varphi} \underline{\mathbb{Z}} \to 0,$$

define $\mathcal{F} := (U \mapsto \varphi(U)^{-1}(1))$. Show that $\mathcal{F}$ is a $\mathcal{G}$-torsor and that isomorphic extensions give isomorphic torsors.

(5) Show that the map from $\mathrm{Ext}^1(\underline{\mathbb{Z}}, \mathcal{G})$ to the set of isomorphism classes of $\mathcal{G}$-torsors defined in (4) is a bijection.

In conclusion, we see that $H^1(X, \mathcal{G})$ can be identified with the set of isomorphism classes of $\mathcal{G}$-torsors. Since $\mathcal{G}$-torsors also make sense for sheaves of not necessarily Abelian groups, this gives a way of defining H^1 for non-Abelian sheaves. [1]

[1] However, in this case, H^1 will only be a pointed set, and not a group.