Dr. Gebhard Martin
Dr. Yajnaseni Dutta

Summer term 2023

Exercises, Algebraic Geometry II – Week 4

Exercise 16. *Local cohomology* (4 points)
Let A be a Noetherian ring and $\mathfrak{a} \subseteq A$ an ideal.

(1) Recall the definition of $\Gamma_{\mathfrak{a}}(-)$ from Exercise 41 of Algebraic Geometry 1. Show that $\Gamma_{\mathfrak{a}}(-) : \mathrm{Mod}_A \to \mathrm{Mod}_A$ is left-exact. Let $H^i_{\mathfrak{a}}(-)$ be its i-th right-derived functor.

(2) Let $X = \operatorname{Spec} A$ and $Y = V(\mathfrak{a})$. Show that, for any A-module M, there exists a natural isomorphism
$$H^i_Y(X, \widetilde{M}) \cong H^i_{\mathfrak{a}}(M).$$

(3) Show that, for every i, $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$.

Exercise 17. *Depth and local cohomology* (4 points)
Let A be a Noetherian ring, $\mathfrak{a} \subseteq A$ an ideal, and M a finitely generated A-module. Recall that an *M-regular sequence* is a sequence of elements $a_1, \ldots, a_n \in A$ such that a_i is not a zero divisor on $M/(a_1, \ldots, a_{i-1})M$. The $\mathfrak{a}$-depth $\operatorname{depth}_{\mathfrak{a}}(M)$ of M is the maximal length of an M-regular sequence $x_1, \ldots, x_n$ such that each x_i is in $\mathfrak{a}$. If A is a local ring with maximal ideal $\mathfrak{m}$, then the $\mathfrak{m}$-depth of A is simply called *depth* and denoted by $\operatorname{depth}(A)$.

(1) Show that $\operatorname{depth}_{\mathfrak{a}}(M) \geq 1$ if and only if $\Gamma_{\mathfrak{a}}(M) = 0$.

(2) Fix $n \geq 0$. Show that $\operatorname{depth}_{\mathfrak{a}}(M) \geq n$ if and only if $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$.

(3) Let X be a Noetherian scheme and $x \in X$ a closed point. Show that the following are equivalent:

   (a) $\operatorname{depth} \mathcal{O}_{X,x} \geq 2$.

   (b) For every open neighborhood U of x in X, every section of $\mathcal{O}_X$ over $U \setminus \{x\}$ extends uniquely to a section over U.

Exercise 18. *Euler characteristics and Riemann–Roch for curves* (4 points)
Let X be a projective scheme over a field k and $\mathcal{F}$ a coherent sheaf on X. The *Euler characteristic* of $\mathcal{F}$ is defined as
$$\chi(\mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

(1) Show that if
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
is a short exact sequence of coherent sheaves on X, then $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$.

Due 05.05.2023, 2pm

(2) Show that if X is a smooth, complete, geometrically integral curve over a perfect field k, and if $\mathcal{L} \in \mathrm{Pic}(X)$, then

$$\chi(\mathcal{L}) = \deg(\mathcal{L}) + \chi(\mathcal{O}_X).$$

(Hint: Show that for every closed point $P \in X$, the formula holds for $\mathcal{L}$ if and only if it holds for $\mathcal{L}(P)$.)

((2) is really the Riemann–Roch formula we know from AG 1. Indeed, Serre duality, which we will prove later, implies the existence of an isomorphism $H^1(X, \mathcal{L}) \cong H^0(X, \mathcal{L}^\vee \otimes \omega_X)^\vee$.)

Exercise 19. *Complete intersections* (4 points)
Let k be a field. A closed subscheme $X \subseteq \mathbb{P}^n_k$, $n \geq 2$, is called (global) *complete intersection*, if every irreducible component of X has the same dimension d and X is the (scheme-theoretic) intersection of $(n-d)$ hypersurfaces. Show that such a complete intersection X satisfies the following properties:

(1) For all $m \in \mathbb{Z}$, the natural map

$$H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m)) \to H^0(X, \mathcal{O}_X(m))$$

is surjective.

(2) $H^i(X, \mathcal{O}_X(m)) = 0$ for all $0 < i < d$.

(3) X is connected.

(4) $\chi(\mathcal{O}_X) = 1 + (-1)^d \dim_k H^d(X, \mathcal{O}_X)$.

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 20. *Chow's lemma* (+ 4 extra points)
The goal of this exercise is to prove *Chow's lemma*:

Let $X \to S$ be a proper morphism with S Noetherian. Then, there is a scheme X' and a morphism $g : X' \to X$ such that $X' \to X \to S$ is projective and there is an open dense subset $U \subseteq X$ such that $g^{-1}(U) \to U$ is an isomorphism.

(1) Reduce to the case where X is irreducible.

(2) Show that X can be covered by open subsets $U_1, \dots, U_n$ that are quasi-projective over S. Let $U_i \to P_i$ be the corresponding open immersions into schemes P_i that are projective over S.

(3) Let $U = \bigcap_{i=1}^{n} U_i$ and consider the map

$$f : U \to X \times_S P_1 \times_S \dots \times_S P_n$$

induced by the inclusion $U \hookrightarrow X$ and the maps $U \hookrightarrow U_i \hookrightarrow P_i$. Let X' be the image of f [1]. Let $g : X' \to X$ be the projection to the first factor and $h : X' \to P_1 \times_S \dots \times_S P_n$ be the projection to the remaining factors. Show that h is a closed immersion and deduce that $X' \to S$ is projective.

(4) Conclude by showing that $g^{-1}(U) \to U$ is an isomorphism.

[1] i.e. the smallest closed subscheme of the right-hand side through which f factors