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Exercises, Algebraic Geometry II – Week 5

Exercise 21. *Projection formula* (4 points)
Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Show that, for all i, there exists an isomorphism

$$R^i f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong R^i f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

Exercise 22. *Coherence of Ext sheaves* (4 points)
Let X be a Noetherian scheme and $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(X, \mathcal{O}_X)$.

(1) Assume $\mathcal{F}$ is coherent and $\mathcal{G}$ is quasi-coherent. Show that, for all $\operatorname{Spec} A = U \subseteq X$ open affine and all $i \geq 0$, there is a natural isomorphism

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})|_U \cong \mathrm{Ext}^i_A(\mathcal{F}(U), \mathcal{G}(U))^{\sim}.$$

Conclude that $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent.

(2) Assume additionally that $\mathcal{G}$ is also coherent. Show that $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent.

Exercise 23. *Not enough projectives* (4 points)
Let k be an infinite field and let $X = \mathbb{P}^1_k$.
Show that, in each of the categories $\mathrm{Mod}(X, \mathcal{O}_X), \mathrm{QCoh}(X, \mathcal{O}_X)$, and $\mathrm{Coh}(X, \mathcal{O}_X)$, there exists no projective object $\mathcal{P}$ with a surjection $\mathcal{P} \to \mathcal{O}_X$. In particular, $\mathcal{O}_X$ is not projective.
(Hint: Consider surjections of the form $\mathcal{O}_V \to k(x)$ and $\mathcal{L} \to \mathcal{L} \otimes k(x)$, where $x \in V \subseteq X$ is open and $\mathcal{L} \in \mathrm{Pic}(X)$)

Exercise 24. *Enough locally frees and homological dimension* (4 points)
Let X be a Noetherian scheme such that $\mathrm{Coh}(X, \mathcal{O}_X)$ *has enough locally frees*, i.e., such that every coherent sheaf on X is a quotient of a locally free sheaf of finite rank. In this case, the *homological dimension* $\mathrm{hd}(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on X is the minimal length of a locally free resolution of $\mathcal{F}$. [1]

(1) Show that, for every $\mathcal{G} \in \mathrm{Mod}(X, \mathcal{O}_X)$, the functors $\mathcal{E}xt^i_{\mathcal{O}_X}(-, \mathcal{G})$ form a (contravariant) universal δ-functor from $\mathrm{Coh}(X, \mathcal{O}_X)$ to $\mathrm{Mod}(X, \mathcal{O}_X)$.

(2) Show that $\mathrm{hd}(\mathcal{F}) \leq n$ if and only if $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0$ for all $i > n$ and $\mathcal{G} \in \mathrm{Mod}(X, \mathcal{O}_X)$.

(3) Show that $\mathrm{hd}(\mathcal{F}) = \sup_{x \in X} \mathrm{pd}(\mathcal{F}_x)$, where $\mathrm{pd}(\mathcal{F}_x)$ is the *projective dimension of* $\mathcal{F}_x$, i.e., the minimal length of a projective resolution of the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.

(4) Show that every projective scheme over a Noetherian ring has enough locally frees.

Due 12.05.2023, 2pm
[1] Set $\mathrm{hd}(\mathcal{F}) = \infty$ if there is no finite locally free resolution.

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 25. *An application of Chow's lemma* (+ 4 extra points)
The goal of this exercise is to prove the following statement:

Let $X \to Y$ be a proper morphism of Noetherian schemes. Let $\mathcal{F} \in \mathrm{Coh}(X, \mathcal{O}_X)$. Then, $R^i f_* \mathcal{F} \in \mathrm{Coh}(Y, \mathcal{O}_Y)$.

Prove this as follows:

(1) Reduce to the case where $Y = \mathrm{Spec}\, A$ is affine. By Chow's lemma, there exists a projective A-scheme X' together with a morphism $g : X' \to X$ over Y such that there exists an open $U \subseteq X$ such that $g^{-1}(U) \to U$ is an isomorphism.

(2) Use induction on $\dim(\mathrm{supp}(\mathcal{F}))$ and the Grothendieck spectral sequence

$$E_2^{p,q} = H^p(X, R^q g_*(g^* \mathcal{F})) \Rightarrow H^{p+q}(X', g^* \mathcal{F})$$

to show that $H^n(X, g_* g^* \mathcal{F})$ is a finitely generated A-module for all n.

(3) Using the same type of induction, conclude that $H^n(X, \mathcal{F})$ is a finitely generated A-module for all n. Conclude that $R^i f_* \mathcal{F} \in \mathrm{Coh}(Y, \mathcal{O}_Y)$.