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Summer term 2023

Exercises, Algebraic Geometry II – Week 6

Exercise 26. *Relative Spec* (4 points)
Let Y be a scheme and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras.

(1) Show that there is a unique morphism of schemes $f : X \to Y$ such that for all open affines $V \subseteq Y$, there exist isomorphisms $f^{-1}(V) \cong \mathrm{Spec}\mathcal{A}(V)$ such that for all open affines $U \subseteq V \subseteq Y$, the inclusion $f^{-1}(U) \subseteq f^{-1}(V)$ is induced by the restriction map $\mathcal{A}(V) \to \mathcal{A}(U)$. We write $X = \underline{Spec}_Y \mathcal{A}$.

(2) Show that a morphism of schemes $f : X \to Y$ is affine if and only if $X \cong \underline{Spec}_Y f_*\mathcal{O}_X$ and f is the map constructed in (1).

(3) Let $f : X \to Y$ be an affine morphism. Show that f_* induces an equivalence of categories between $\mathrm{QCoh}(X, \mathcal{O}_X)$ and the category of $f_*\mathcal{O}_X$-modules that are quasi-coherent as $\mathcal{O}_Y$-modules.

Exercise 27. *Hodge numbers of projective space* (4 points)
Let k be a field. Show that

$$h^{p,q}(\mathbb{P}^n_k) = \begin{cases} k \text{ if } p = q, 0 \leq p, q \leq n \\ 0 \text{ else.} \end{cases}$$

(Clarify to yourself how hard it is to find a smooth projective variety X with these Hodge numbers and such that X is *not* isomorphic to projective space.)

Exercise 28. *Künneth formula* (4 points)
Let X_1 and X_2 be quasi-compact separated schemes over a field k. Let $p_i : X_1 \times_{\mathrm{Spec}\, k} X_2 \to X_i$ be the projection. For $\mathcal{F}_i \in \mathrm{QCoh}(X_i, \mathcal{O}_{X_i})$, we define $\mathcal{F}_1 \boxtimes \mathcal{F}_2 := p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2$.

(1) Show that, for all $n \in \mathbb{Z}$, there is an isomorphism

$$H^n(X_1 \times_{\mathrm{Spec}\, k} X_2, \mathcal{F}_1 \boxtimes \mathcal{F}_2) \cong \bigoplus_{i+j=n} H^i(X_1, \mathcal{F}_1) \otimes_k H^j(X_2, \mathcal{F}_2).$$

(Hint: Use Čech cohomology and the Eilenberg–Zilber theorem)

(2) Assume that X_1 and X_2 are smooth and projective over a perfect k. Show that

$$\omega_{X_1 \times_{\mathrm{Spec}\, k} X_2} \cong \omega_{X_1} \boxtimes \omega_{X_2}.$$

Express $h^{p,q}(X_1 \times_{\mathrm{Spec}\, k} X_2)$ in terms of the Hodge numbers of X_1 and X_2.

(3) Calculate all Hodge numbers $h^{p,q}(\mathbb{P}^1 \times_{\mathrm{Spec}\, k} C)$, where C is a smooth complete geometrically integral curve over a perfect k.

Due 19.05.2023, 2pm

Exercise 29. *Arithmetic genus and singular curves* (4 points)
Let X be a geometrically integral curve over a perfect field k. Let $f : \widetilde{X} \to X$ be the normalization of X. The *arithmetic genus* of X is defined as $p_a(X) = 1 - \chi(\mathcal{O}_X)$. Show the following:

(1) If X is smooth, then $p_a(X) = g(X)$.

(2) There exists a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{\widetilde{X}} \to \bigoplus_{x \in X} (f_*\mathcal{O}_{\widetilde{X}})_x/\mathcal{O}_{X,x} \to 0.$$

Let $\delta_x = \mathrm{length}_{\mathcal{O}_{X,x}} (f_*\mathcal{O}_{\widetilde{X}})_x/\mathcal{O}_{X,x}$.

(3) $p_a(X) = p_a(\widetilde{X}) + \sum_{x \in X} \delta_x$.

(4) If $p_a(X) = 0$, then X is smooth and in fact $X \cong \mathbb{P}^1_k$.

(5) If X is a plane curve of degree d, then X has at most $\frac{1}{2}(d-1)(d-2)$ singular points.

(6) Calculate δ_x for all points $x \in X$, where X is one of the following planar curves: a) $V_+(y^2z - x^3)$ b) $V_+(y^2z - x^2z - x^3)$.

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 30. *A scheme that is not Cohen–Macaulay* (+ 4 extra points)
Let k be a field and $X = \mathrm{Spec}\, A$ with $A = k[x, y, z, w]/(wy, wz, xy, xz)$, i.e., X is the union of two 2-planes in $\mathbb{A}^4_k$ meeting in the origin x. Show that $\mathrm{depth}(\mathcal{O}_{X,x}) = 1$. Conclude that X is not (S_2), hence not Cohen–Macaulay and not normal, even though it is (R_1).