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Exercises, Algebraic Geometry II – Week 7

Exercise 31. *Fiber dimension and flat pullback* (4 points)
Show the following statements:

(1) If X is an equidimensional scheme of dimension n locally of finite type over a field k and $k \subseteq K$ is any field extension, then X_K is equidimensional of dimension n. Conclude that "locally of finite type and flat of relative dimension n" satisfies (BC).

(2) Let $f : X \to Y$ be a morphism of schemes. Let $Z \subseteq Y$ be a closed subscheme with ideal sheaf $\mathcal{I}_Z$ and let $\mathcal{I}_{X \times_Y Z}$ be the ideal sheaf of the closed subscheme $X \times_Y Z \subseteq X$. Show that there is a natural map $f^*\mathcal{I}_Z \to \mathcal{I}_{X \times_Y Z}$ which is an isomorphism if f is flat.

(For the following two exercises, you may use the flatness criteria of Lecture 12)

Exercise 32. *Flat and non-flat morphisms* (4 points)
Decide whether the following morphisms are flat.

(1) The natural projection $f\colon X \to \operatorname{Spec}(\mathbb{Z})$ where $X = V(3x^2 + 6y^2) \subseteq \mathbb{A}^2_{\mathbb{Z}}$.

(2) The natural projection $f\colon X \to \operatorname{Spec}(\mathbb{Z})$ where $X \subseteq \mathbb{A}^2_{\mathbb{Z}}$ is the closure (with reduced scheme structure) $V(3x^2 + 6y^2) \subseteq \mathbb{A}^2_{\mathbb{Q}}$.

(3) The normalization of $k[x, y]/(xy)$, where k is a field. (Normalization of $X = \bigcup_{i=1}^n X_i$ where X_i's are integral is defined to be $\bigsqcup_{i=1}^n X_i^\nu$ where X_i^ν is the normalization of X_i.)

(4) (+1 extra point) The morphism

$$f\colon \operatorname{Spec} k[x, y] \to \operatorname{Spec} k[x, y, z, w]/(xz + z^2, yz + wz, xw + zw, yw + w^2),$$

where k is a field.

Exercise 33. *Hilbert polynomial and universal hypersurface* (4 points)
Let X be a projective scheme over a field k, let $\mathcal{O}_X(1)$ be a very ample sheaf on X, and let $\mathcal{F} \in \operatorname{Coh}(X, \mathcal{O}_X)$.

(1) Show that there exists a polynomial $P_{\mathcal{F}} \in \mathbb{Q}[x]$ such that $P_{\mathcal{F}}(n) = \chi(\mathcal{F}(n))$ for all $n \in \mathbb{Z}$. This polynomial is called *Hilbert polynomial of $\mathcal{F}$ (with respect to $\mathcal{O}_X(1)$)*. If $\mathcal{F} = \mathcal{O}_X$, we call it the Hilbert polynomial of X.

(Hint: Use I.7.3. of Hartshorne's book and induction on the dimension of the support of $\mathcal{F}$.)

(2) Calculate the Hilbert polynomial of $\mathbb{P}^n_k$ with respect to $\mathcal{O}(1)$.

(3) Calculate the Hilbert polynomial of hypersurfaces of degree d.

(4) Let $\mathcal{X} \subseteq \mathbb{P}^n_k \times \mathbb{P}^N_k$ be the "incidence variety" of all degree d hypersurfaces in $\mathbb{P}^n$ given by the equation $\sum_I a_I x^I$, where $x^I = x_0^{i_0} \dots x_n^{i_n}$ and the sum is over all $I = (i_0, \dots, i_n)$ with $\sum i_k = d$ and $N = \dim_k \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) - 1$. Show that the projection $\mathcal{X} \to \mathbb{P}^N_k$ is flat.

Due 26.05.2023, 2pm

Exercise 34. *Hilbert polynomials of curves* (4 points)
Let k be an algebraically closed field, let $n \geq 2$, and consider a smooth geometrically integral projective curve $C \subseteq \mathbb{P}^n_k$. Let P_C be the Hilbert polynomial of C with respect to the given embedding of C.

(i) Show that $P_C(x) = dx + 1 - g(C)$, where d is the degree of $\mathcal{O}_C(1)$.

(ii) By a theorem of Bertini (which will be proved in the lecture soon), there exists a hyperplane $H \subseteq \mathbb{P}^n_k$ such that $C \cap H$ is a finite reduced scheme, or, equivalently, there exists a section $s \in H^0(\mathbb{P}^n_k, \mathcal{O}(1))$ that yields a short exact sequence of sheaves on C

$$0 \to \mathcal{O}_C \xrightarrow{\cdot s} \mathcal{O}_C(1) \to \bigoplus_{i=1}^{r} k(p_i) \to 0.$$

for some $r > 0$, where the p_i are pairwise distinct closed points and $k(p_i)$ is the skyscraper sheaf with value k at p_i. Show that $r = d$.

(In other words, d is the number of intersection points of C with a general hyperplane.)

(iii) Show that the map $H^0(C, \mathcal{O}_C(m)) \to k^{\oplus d}$ induced by the above sequence is surjective for $m \geq d - 1$.

(iv) Deduce that $H^1(C, \mathcal{O}_C(d - 2)) = 0$ and use this to show that

$$g(C) \leq d^2 - 2d$$

if $d \geq 2$.

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 35. *Miracle Flatness* (+ 4 extra points)
Recall that for a flat morphism $f : X \to Y$ between equidimensional k-schemes of finite type, the dimension of the fibers of f is exactly the difference of the dimensions of X and Y. Prove the following converse to this statement, called *Miracle flatness*:

Let $f : X \to Y$ be a morphism between equidimensional k-schemes of finite type. Assume that X is Cohen–Macaulay, Y is regular, and the fibers of f have dimension $\dim(X) - \dim(Y)$. Then, f is flat.