Dr. Gebhard Martin
Dr. Yajnaseni Dutta

Summer term 2023

Exercises, Algebraic Geometry II – Week 8

Exercise 36. *Flatness is an open property* (+ 4 extra points)
Let $f : X \to Y$ be a morphism of finite type between Noetherian schemes. Show that the locus

$$\{x \in X \mid \ f \text{ is flat at } x\} \subseteq X$$

is open (possibly empty).

Exercise 37. *Codimension of the non-flat locus for finite morphisms* (4 points)
Let $f : X \to Y$ be a finite surjective morphism between integral Noetherian schemes.

(1) Let $V \subseteq Y$ be the set of points over which f is flat. Show that V is open.

(2) Assume that Y is regular in codimension 1. Show that $\operatorname{codim}(Y \setminus V, Y) \geq 2$.

(3) Show that (2) fails in general if Y is not assumed to be regular in codimension 1.

(4) Show that if f is not finite, then the maximal subset $V \subseteq Y$ over which f is flat is not necessarily open.

Exercise 38. *Flat limits* (4 points)
Let A be a discrete valuation ring, let $0 \in \operatorname{Spec} A$ be the closed point, and let $\eta \in \operatorname{Spec} A$ be the generic point. Let X be a Noetherian scheme over $\operatorname{Spec} A$ and let $Y \subseteq X_\eta$ be a closed subscheme.

(1) Show that the scheme-theoretic closure $\overline{Y}$ of Y in X (i.e. the smallest closed subscheme of X containing Y) is flat over $\operatorname{Spec} A$.

(2) Show that if $Y' \subseteq X$ is another closed subscheme which is flat over A and restricts to Y over the generic point, then $Y' = \overline{Y}$. The special fiber $(\overline{Y})_0$ is called *the flat limit* of Y at 0.

(3) Let $A = k[t]_{(t)}$, let $X = \mathbb{A}^3_A$, and $Y = V(y, z) \cup V(x, z - t)$. Compute the flat limit of Y at $t = 0$. Draw a picture!

Exercise 39. *First order deformations* (4 points)
Let $Y \subseteq X$ be a closed subscheme of a scheme X of finite type over a field k. Let $D = k[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers. A *first order deformation of* Y *in* X is a closed subscheme $Y' \subseteq X \times_{\operatorname{Spec} k} \operatorname{Spec} D$ which is flat over D and whose closed fiber is Y.

(1) Show that the set of first order deformations of Y in X is in bijection with $H^0(Y, \mathcal{N}_{Y/X})$, where $\mathcal{N}_{Y/X} := (\mathcal{I}/\mathcal{I}^2)^\vee$ is the normal sheaf of Y in X, where $\mathcal{I}$ is the ideal of Y in X.

(2) Assume that Y is a geometrically integral smooth projective curve on a smooth surface X over a perfect field k and that $\mathcal{N}_{Y/X}$ has negative degree on Y. Show that Y has no non-trivial first order deformations in X (we say that Y is (infinitesimally) *rigid* in X).

Due 09.06.2023, 2pm

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 40. *Some more deformation theory* (+ 4 extra points)
Let A be a finitely generated algebra over a field k. Choose an exact sequence

$$0 \to I \to k[x_1, \ldots, x_n] \to A \to 0,$$

consider the associated conormal sequence

$$I/I^2 \to \Omega_{k[x_1,\ldots,x_n]/k} \to \Omega_{A/k} \to 0,$$

and apply $\mathrm{Hom}_A(-, A)$ to obtain

$$0 \to \mathrm{Hom}_A(\Omega_{A/k}, A) \to \mathrm{Hom}_A(\Omega_{k[x_1,\ldots,x_n]/k}, A) \to \mathrm{Hom}_A(I/I^2, A) \to T^1(A) \to 0,$$

where $T^1(A)$ is defined via this sequence.

(1) Show that $T^1(A)$ is in bijection with the set of isomorphism classes of first order deformations of A, i.e., flat D-algebras $\mathcal{A}$ together with an isomorphism $\mathcal{A} \otimes_D k \cong A$.

(2) Conclude that $T^1(A)$ is independent of the presentation of A as a quotient of a polynomial ring.

(3) Conclude that if A is smooth over k, then A has no non-trivial first order deformations.

Reflex test

1. Let X be a scheme. For which points $x \in X$ is $\mathrm{Spec}(k(x)) \to X$ a flat morphism?
2. Are there varieties that are neither projective nor quasi-projective nor affine?
3. Describe an example of a birational morphism $f\colon X \to Y$ whose image is neither open nor closed.
4. Write down an example of a field extension $K_1 \subseteq K_2$ with K_2/K_1 algebraic but $\Omega_{K_2/K_1} \neq 0$.
5. Let A be a k-algebra. Compare $\Omega_{k[x_1,\dots,x_n]/k}$ with $\Omega_{A[x_1,\dots,x_n]/A}$.
6. Consider morphisms of schemes $f\colon X \to Y$ and $g\colon Y \to Z$. Is the natural morphism $f^*\Omega_{Y/Z} \to \Omega_{X/Y}$ always injective?
7. Let X be an irreducible scheme of finite type over a field k. Is X smooth over k if $\Omega_{X/k}$ is locally free?
8. Let X be an integral scheme of finite type over a field k and $x \in X$. Compare $\dim_{K(X)} \Omega_{K(X)/k}$ and $\dim_{k(x)}(\Omega_{X/k} \otimes k(x))$.
9. Describe the modules $\Omega_{k/k}$, $\Omega_{k[x]/(x^2)/k}$ and $\Omega_{k(x)/k}$.
10. Find an example of a non-empty, integral, finite type k-scheme X for which there exists no non-empty open subset $U \subset X$ which is smooth over k.
11. Is there a natural map $d\colon \mathcal{O}_X \to \Omega_{X/k}$?
12. Is a flat morphism always surjective? Can the image of a flat morphism be closed?
13. What is the canonical bundle $\omega_{X/k}$ of $X = \mathbb{P}^n \times_k \mathbb{P}^m$?
14. Is there any relation between $\dim(X)$, $\mathrm{rk}(\Omega_{X/k})$, and $\mathrm{trdeg}(K(X)/k)$?
15. What causes the problems when one wants to compare the Zariski tangent space $T_{X,x}$ and the fibre $\Omega_{X/k} \otimes k(x)$?
16. Give an example of a regular ring and of a non-regular ring. What is the easiest/standard example of a regular non-smooth k-scheme? Can you think of one other example?
17. What is the Hilbert polynomial of a reduced k-scheme consisting of two points? Does the answer depend on the field k or on the projective embedding?
18. Give examples of modules that are flat and of those that are not over the two ring $k[x_1, x_2]$ and $k[x]/x^3$.
19. Are the function fields of the Fermat curves $V_+(x_0^d + x_1^d + x_2^d) \subset \mathbb{P}^2$ isomorphic for all d?
20. Can $\omega_{X/k}$ be a trivial invertible sheaf without $\Omega_{X/k}$ being a trivial locally free sheaf?
21. Which of the following properties of a scheme over a field are preserved by passing to a field extension: regular, integral, irreducible, reduced? What if the field has characteristic 0?
22. Which of the following properties of a flat morphism f can you check by checking it on fibers over closed points: proper, finite? Can you also do that without assuming that f is flat?
23. Is non-flatness always detected by a jump in fiber dimension?
24. Recall the two standard right exact sequences involving $\Omega_{B/A}$ and the corresponding sheaf versions.
25. How do you pass from the ideal sheaf of the diagonal $\Delta\colon X \to X \times_Y X$ to $\Omega_{X/Y}$?
26. Have you memorized the Euler sequence?
27. When is a scheme X flat over a field k?
28. Which of the following morphisms are flat? a) The blow-up of the origin in $\mathbb{A}^2$. b) A closed embedding. c) An open embedding. d) A finite morphism between smooth projective curves.
29. Explain why there are always maps $Ext^p(\mathcal{F}, \mathcal{G}) \to H^0(X, \mathcal{E}xt^p(\mathcal{F}, \mathcal{G}))$ and $H^p(X, \mathcal{F}) \to H^0(Y, R^p f_* \mathcal{F})$
30. For what general conditions on X and $\mathcal{F}$ do we have $H^i(X, \mathcal{F})$ finite-dimensional vector space?
31. Let D be a hypersurface in $\mathbb{P}^n$. What is $H^i(X \setminus D, \mathcal{O}_{X \setminus D})$?
32. Let $X_1, X_2 \subseteq \mathbb{P}^n_k$ be projective varieties that are isomorphic as schemes. Do the Hilbert polynomials of X_1 and X_2 agree?
33. State general conditions on X, $\mathcal{F}$ and i such that $H^i(X, \mathcal{F}) = 0$.
34. Under which assumptions on a scheme X over a field k does Serre duality hold on X and in what sense?
35. Do you remember all right-derived functors that appeared in the lecture?