Dr. Gebhard Martin
Dr. Yajnaseni Dutta

Summer term 2023

Exercises, Algebraic Geometry II – Week 9

Exercise 41. *Smooth and non-smooth morphisms* (4 points)
Let k be a field. Consider the projection $\pi \colon \mathbb{A}^2_k \to \mathbb{A}^1_k$, $x \mapsto x_1 + x_2$. Decide whether the restriction of π to $X \subseteq \mathbb{A}^2_k$ is flat or smooth, where X is:

(1) $X = V(x_1^2 - x_2^2)$;

(2) $X = V(x_1^2 + x_2^2 + 2x_1x_2 - x_2 + x_1)$;

(3) $X = \mathbb{A}^2_k \setminus V(x_1 - x_2)$;

(4) $X = V((x_1 - x_2)(x_1 - 1), (x_1 - x_2)(x_1 + x_2))$.

Exercise 42. *Étale covering of the node* (4 points)
Let k be a field of characteristic different from 2. Consider the affine nodal cubic curve $C \coloneqq V(x_2^2 - x_1^2(x_1 + 1)) \subseteq \mathbb{A}^2_k$. Construct a curve C' together with a finite étale morphism $f \colon C' \to C$ such that C' has two irreducible components and the restriction of f to each of them describes the normalization of C. Draw a picture!

(Hint: Use the map $\mathbb{A}^2_k \to \mathbb{A}^2_k$ given by $x_1 \mapsto y_2^2 - 1, x_2 \mapsto y_1y_2$.)

Exercise 43. *Infinitesimal lifting criteria* (4 points)
A closed immersion of schemes $S_0 \hookrightarrow S$ with ideal sheaf $\mathcal{I}$ is called *first order thickening* if $\mathcal{I}^2 = 0$. A morphism of schemes $f : X \to Y$ is called *formally smooth* (resp. *formally unramified*, resp. *formally étale*) if for all commutative diagrams of solid arrows

$$\begin{tikzcd} S_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

where $S_0 \to S$ is a first order thickening, some (resp. at most one, resp. a unique) dotted arrow making the diagram commute exists. Now, assume that $X = \operatorname{Spec} B, Y = \operatorname{Spec} A, S = \operatorname{Spec} C$, and $S_0 = \operatorname{Spec} C/I$ for some ideal I with $I^2 = 0$.

(1) Let $\phi_1, \phi_2 : B \to C$ be two morphisms inducing a dotted arrow making the diagram commute. Show that the map $\phi_1 - \phi_2$ induces an A-linear derivation $B \to I$.

(2) Assume that $\phi : B \to C$ is a morphism inducing a dotted arrow making the diagram commute and $D : B \to I$ is an A-linear derivation. Show that $\phi + D : B \to C$ induces another dotted arrow making the diagram commute.

(3) Show that f is formally unramified if and only if $\Omega_{B/A} = 0$.

(Hint: For a B-module M, consider $B[M] = \{b + m \mid b \in B, m \in M\}$.)

Due 16.06.2023, 2pm

(4) (+2 extra points) Show that if f is formally smooth, then $\Omega_{B/A}$ is a projective B-module.

(One can show that a morphism of schemes is smooth/unramified/étale as defined in the lecture if and only if it is locally of finite presentation and formally smooth/unramified/étale. This goes under the name "infinitesimal lifting criterion". Compare the definition of formal smoothness with the definition of submersion in differential geometry.)

Exercise 44. *Étale morphisms* (4 points)
Decide which of the following morphisms is étale or at least unramified.

(1) $\mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k \setminus \{0\}$, $t \mapsto t^2$.

(2) $\mathbb{P}^n_k \to \mathbb{P}^n_k$, $[x_0 : \cdots : x_n] \mapsto [x_0^\ell : \cdots : x_n^\ell]$ where $l > 1$.

(3) $\mathrm{Spec}(\mathbb{Q}(\sqrt{5})) \to \mathrm{Spec}(\mathbb{Q})$.

(4) $\mathrm{Spec}(k[t]) \to \mathrm{Spec}(k[x,y]/(x^3 - y^2))$ given by $x \mapsto t^2$, $y \mapsto t^3$.

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 45. *A strange net of plane cubics* (+ 4 extra points)
Let k be an algebraically closed field of characteristic 2. Let $S \subseteq \mathbb{P}^2_k$ be the subset of seven points with coordinates in $\mathbb{F}_2$. Consider the linear system L of plane cubic curves that pass through the seven points of S.

(1) Show that L determines a morphism $\mathbb{P}^2_k \setminus S \to \mathbb{P}^2_k$ which is generically finite of degree 2 and such that the corresponding extension of function fields is purely inseparable.

(2) Show that every curve C in L is singular. More precisely, show that C is either a union of lines or an irreducible cubic with a cusp (i.e. C is projectively equivalent to $V_+(y^2 z + x^3)$).

(3) Show that the singularities of the curves in L cover $\mathbb{P}^2_k$.