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Dr. Yajnaseni Dutta Winter term 2022/23

Exercises, Algebraic Geometry I – Week 1

Exercise 1. *Direct sum of sheaves* (3 points)
Let $\mathcal{F}$, $\mathcal{G}$ be two sheaves of abelian groups on a topological space X. Show that $\mathcal{F} \oplus \mathcal{G} \colon U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ defines a sheaf.

Exercise 2. *Sheaf $\mathcal{H}om$* (4 points)
For a (pre-)sheaf $\mathcal{F}$ on a topological space X and an open subset $U \subset X$ one defines the restriction $\mathcal{F}|_U$ to be the (pre-)sheaf on the topological space U given by $\mathcal{F}|_U(V) \coloneqq \mathcal{F}(V)$ for any open subset $V \subset U$. Then for two sheaves $\mathcal{F}, \mathcal{G}$ of abelian groups, $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ denotes the abelian group of all sheaf homomorphisms $\mathcal{F}|_U \to \mathcal{G}|_U$. Show that this naturally defines a pre-sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \colon U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ which is in fact a sheaf.

Exercise 3. *Gluing of sheaves* (5 points)
Let X be a topological space and let $X = \bigcup U_i$ be an open covering. We use the shorthand $U_{ij} \coloneqq U_i \cap U_j$ and $U_{ijk} \coloneqq U_i \cap U_j \cap U_k$.
Consider sheaves of abelian groups $\mathcal{F}_i$ on U_i and isomorphisms (*gluings*) $\varphi_{ij} \colon \mathcal{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j|_{U_{ij}}$. Show that if the *cocycle condition* $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on U_{ijk} is satisified, then there exists a sheaf of abelian groups $\mathcal{F}$ on X together with isomorphisms $\varphi_i \colon \mathcal{F}|_{U_i} \cong \mathcal{F}_i$ such that $\varphi_{ij} \circ \varphi_i = \varphi_j$ on U_{ij}. The sheaf $(\mathcal{F}, \varphi_i)$ is unique up to unique isomorphism.

Exercise 4. *Exponential map* (4 points)
Consider $X = \mathbb{C} \setminus \{0\}$ with its usual topology and let $\mathcal{O}_X$ be the sheaf of holomorphic functions, i.e. $\mathcal{O}_X(U) = \{f \colon U \to \mathbb{C} \mid f \text{ holomorphic}\}$. Similarly, let $\mathcal{O}_X^*$ be the sheaf of holomorphic functions without zeroes. (Throughout, you may work with differentiable functions instead of holomorphic ones if you prefer.)

Show that the exponential map defines a morphism of sheaves (of abelian groups)

$$\exp : \mathcal{O}_X \to \mathcal{O}_X^*, \mathcal{O}_X(U) \ni f \mapsto \exp(f) \in \mathcal{O}_X^*(U).$$

Find a basis of the topology on X such that $\exp_U : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$ is surjective for all U in this basis. Note that $\mathcal{O}_X(X) \to \mathcal{O}_X^*(X)$ is not surjective. Describe the kernel of $\exp_U$.

The following exercise uses stalks, which will be introduced in the second lecture.

Exercise 5. *'Espace étalé' of a presheaf* (4 points)
Let $\mathcal{F}$ be a presheaf of sets on a topological space X. Define the set

$$|\mathcal{F}| \coloneqq \bigsqcup_{x \in X} \mathcal{F}_x,$$

which comes with a natural projection $\pi : |\mathcal{F}| \to X$, $(s \in \mathcal{F}_x) \mapsto x$. Then any $s \in \mathcal{F}(U)$ defines a section of π over U by $x \mapsto s_x$. One endows $|\mathcal{F}|$ with the strongest topology such that all $s \in \mathcal{F}(U)$ define continuous sections $x \mapsto s_x$. Show that the sheafification $\mathcal{F}^+$ can be described as the sheaf of continuous sections of $|\mathcal{F}| \to X$.

Due 17.10.2022, 8am

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 6. *Grothendieck topologies* (4 extra points)
The notion of a (pre-)sheaf on a topological space can be formalized as follows: A *Grothendieck topology* $(\mathcal{C}, \mathrm{Cov}_{\mathcal{C}})$ consists of a category $\mathcal{C}$ with a set $\mathrm{Cov}_{\mathcal{C}}$ of collections $\{\pi_i : U_i \to U\}_i$ of morphisms in $\mathcal{C}$ (called *coverings* of U) subject to the following conditions:

(1) Any isomorphism $\varphi : V \xrightarrow{\sim} U$ defines a covering $\{\varphi : V \to U\} \in \mathrm{Cov}_{\mathcal{C}}$.

(2) Suppose we are given $\{\pi_i : U_i \to U\}_i \in \mathrm{Cov}_{\mathcal{C}}$ and for each i a covering $\{\pi_{ij} : U_{ij} \to U_i\}_j \in \mathrm{Cov}_{\mathcal{C}}$. Then $\{\pi_i \circ \pi_{ij} : U_{ij} \to U\}_{ij} \in \mathrm{Cov}_{\mathcal{C}}$ is a covering.

(3) If $\{\pi_i : U_i \to U\}_i$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then $\{\tilde{\pi}_i : U_i \times_U V \to V\}_i$ is a covering.

(In particular, one assumes that the fibre products in (3) exist. Recall the abstract notion of a fibre product.)

(i) Show that for a topological space X the category of open sets Ouv_X comes with a natural Grothendieck topology given by the usual open coverings $U = \bigcup U_i$. Show that the notions presheaf, sheaf, stalk, morphism of (pre)sheaves, etc., can be phrased entirely in terms of this Grothendieck topology.

(ii) For a finite group G consider the category G-*Sets* of sets S with a left G-action $G \times S \to S$. Morphisms in this category are G-equivariant maps, i.e. maps that commute with the G-action. Show that the collections of $\{S_i \to S\}_i$ with $\bigcup S_i \to S$ surjective define a Grothendieck topology on G-*Sets*.

(iii) The group G itself comes with natural left and right G-actions (by multiplication). The corresponding object is denoted $\langle G \rangle \in G\text{-}Sets$. Show that any sheaf $\mathcal{F}$ of sets on G-*Sets* yields a set $\mathcal{F}(\langle G \rangle)$ that is endowed with a natural left G-action. (In fact, $\mathcal{F}$ is determined by this G-set, as $\mathcal{F}(S) = \mathrm{Hom}_{G\text{-}Sets}(S, \mathcal{F}(\langle G \rangle))$. This association defines an equivalence of categories (Sheaves on G-*Sets*) $\xrightarrow{\sim}$ G-*Sets*.)

(iv) The final object in G-*Sets* consists of a set $\{*\}$ of one element. Show that for a sheaf $\mathcal{F}$ of sets the space of sections $\mathcal{F}(\{*\})$ is the fixed point set $\mathcal{F}(\langle G \rangle)^G$.