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Winter term 2022/23

Exercises, Algebraic Geometry I – Week 2

Exercise 7. *Sheafification* (3 points)
Describe examples of presheaves (of abelian groups) $\mathcal{F}$ for which the sheafification $\mathcal{F} \to \mathcal{F}^+$ is not injective resp. not surjective on some open set. Find an example with $\mathcal{F} \neq 0$ but $\mathcal{F}^+ = 0$.

Exercise 8. *Subsheaf with support and left-exactness* (6 points)
Let $\mathcal{F}$ be a sheaf on a topological space X and let $Z \subseteq X$ be a closed subset.

(i) For $U \subseteq X$ open and $s, t \in \mathcal{F}(U)$, show that the set of points $x \in U$ with $s_x = t_x$ in $\mathcal{F}_x$ is open in U. In particular, if $\mathcal{F}$ is a sheaf of abelian groups, then the *support of s* defined as

$$\mathrm{supp}(s) = \{x \in U \mid s_x \neq 0\}$$

is closed in U.

(*Warning:* Observe that in the situation of Exercise 9 for $f \in \mathcal{C}^0(U)$ the zero locus does not necessarily coincide with $U \setminus \mathrm{supp}(f)$.)

(ii) Show that the association

$$\mathcal{H}^0_Z(\mathcal{F}) : U \mapsto \Gamma_{Z\cap U}(U, \mathcal{F}) := \{s \in \mathcal{F}(U) \mid \mathrm{supp}(s) \subseteq Z \cap U\}$$

defines a subsheaf of $\mathcal{F}$.

(A sheaf $\mathcal{F}$ of abelian groups is said to be *supported on Z* if $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{F}$.)

(iii) Prove that $\Gamma_{Z\cap U}(U, \ ) \colon \mathrm{Sh}(X) \to (Ab)$ is a left exact functor, i.e. for any short exact sequence of sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

the sequence

$$0 \to \Gamma_{Z\cap U}(U, \mathcal{F}_1) \to \Gamma_{Z\cap U}(U, \mathcal{F}_2) \to \Gamma_{Z\cap U}(U, \mathcal{F}_3)$$

is exact. Note the important special case $Z = X$, where $\Gamma_{Z\cap U}(U, -) = \Gamma(U, -)$.

(*Warning:* But usually $\Gamma_{Z\cap U}(\mathcal{F}_2) \to \Gamma_{Z\cap U}(\mathcal{F}_3)$ is not surjective, i.e. $\Gamma_{Z\cap U}$ is not exact.)

Exercise 9. *Local rings of continuous functions* (3 points)
Let X be a topological space and let $\mathcal{C}^0$ be the sheaf of continuous functions on X. Consider for a point $x \in X$ the stalk $\mathcal{C}^0_x$. Show that the map $\mathcal{C}^0_x \to \mathbb{R}$, $f \mapsto f(x)$ is well defined and that $\mathcal{C}_x$ is a local ring with maximal ideal $\mathfrak{m}_x := \{f \in \mathcal{C}^0_x \mid f(x) = 0\}$. Describe similar situations involving differentiable or holomorphic functions.

Please turn over

Due **this Friday** 21.10.2022, 2pm

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 10. *Functor of points and the Yoneda lemma* (4 extra points)
Let $\mathcal{C}$ be a category with sets of morphisms between two objects X, Y denoted $\mathrm{Mor}(X, Y)$. Then every object X in $\mathcal{C}$ induces a functor

$$h_X : \mathcal{C}^{\mathrm{op}} \to (\mathit{Sets}),\ Y \mapsto h_X(Y) := \mathrm{Mor}(Y, X).$$

Observe that $h_X(X)$ contains a distinguished element.

(i) Consider the three categories $\mathcal{C} := (\mathit{Top})$ (of topological spaces); $\mathcal{C} := (\mathit{Ab})$ (of abelian groups); $\mathcal{C} := (\mathit{Rings})$ (of rings) and denote for each object X in $\mathcal{C}$ by $|X|$ the underlying set (the set of points). Show that in all three cases there exists an object Z in $\mathcal{C}$ such that for all X the set of points $|X|$ can be recovered as $|X| = h_X(Z)$.

(ii) Consider the category $\mathcal{C} := (\mathit{Rings})^{\mathrm{op}}$. Does there exist an object as in (i) in this case?

(iii) For an arbitrary category $\mathcal{C}$, denote by $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (\mathit{Sets}))$ the category of functors $\mathcal{C}^{\mathrm{op}} \to (\mathit{Sets})$ and consider the functor

$$\begin{aligned} h\colon \quad \mathcal{C} &\to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (\mathit{Sets})) \\ X &\mapsto h_X. \end{aligned}$$

The Yoneda lemma then asserts that h is a fully faithful embedding, in other words h defines an equivalence of categories between $\mathcal{C}$ and a full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (\mathit{Sets}))$. Spell out what this means and try to prove it. Check Vakil's notes on the subject (or any other source). Objects in the image of h (or, more precisely, objects isomorphic to objects in the image) are called *representable functors*.