Dr. Gebhard Martin
Dr. Yajnaseni Dutta

Winter term 2022/23

Exercises, Algebraic Geometry I – Week 3

Exercise 11. *Direct and inverse image are adjoint* (6 points)
Let $f\colon X \to Y$ be a continuous map. Show that $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is right adjoint to $f^{-1}\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ (one writes $f^{-1} \dashv f_*$), i.e. for all $\mathcal{F} \in \mathrm{Sh}(X)$ and $\mathcal{G} \in \mathrm{Sh}(Y)$, there exists an isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F})$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$. Show that, in particular, there exist natural homomorphisms

$$\mathcal{G} \to f_*f^{-1}\mathcal{G} \text{ and } f^{-1}f_*\mathcal{F} \to \mathcal{F}.$$

Verify also that for the composition of two continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$ one has $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Exercise 12. *Stalks of direct image sheaf* (3 points)
Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of abelian groups on X.

(i) Show that if $y \in Y$ is not in the closure of $f(X)$ in Y, then $(f_*\mathcal{F})_y = 0$.

(ii) Show that if f is the inclusion of a subspace (i.e. f is injective and $U \subseteq X$ is open if and only if there exists $V \subseteq Y$ open with $f^{-1}(V) = U$) and $y = f(x)$ for some $x \in X$, then $(f_*\mathcal{F})_y \cong \mathcal{F}_x$.

(iii) Give an example where f is as in (ii), but y is not in $f(X)$, and $(f_*\mathcal{F})_y \neq 0$.

Exercise 13. *Direct image under point inclusion* (3 points)
Let $x \in X$ be an arbitrary point (not necessarily closed) of a topological space X. Is the direct image $(i_x)_*\colon \mathrm{Sh}(\{x\}) \to \mathrm{Sh}(X)$ associated with the inclusion $i_x\colon \{x\} \hookrightarrow X$ exact? Give a proof or a counterexample.

Exercise 14. *Glueing morphisms of locally ringed spaces* (4 points)
Let X and Y be locally ringed spaces. For each $U \subseteq X$ open, let $Hom(U, Y)$ be the set of morphisms $(U, \mathcal{O}_X|_U) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces. Show that $U \mapsto Hom(U, Y)$ defines a sheaf of sets on X.

(In particular: For every open cover $X = \cup_{i \in I} U_i$, giving a morphism of locally ringed spaces $\varphi : X \to Y$ is the same as giving morphisms $\varphi_i : U_i \to Y$ which agree on intersections.)

Please turn over

Due 28.10.2022, 2pm

The following exercise uses the equivalence of categories between the category of affine schemes and the opposite category of the category of rings, which will be introduced in the lecture on Monday.

Exercise 15. *Rational points* (4 points)
Let $(X, \mathcal{O}_X)$ be a scheme and let $x \in X$ with residue field $k(x) \coloneqq \mathcal{O}_{X,x}/\mathfrak{m}_x$.

(i) Show that, for a field K, to give a morphism of schemes $(\mathrm{Spec}(K), \mathcal{O}_{\mathrm{Spec}(K)}) \to (X, \mathcal{O}_X)$ with image x is equivalent to giving a field inclusion $k(x) \hookrightarrow K$.

(ii) If $(X, \mathcal{O}_X)$ is a k-scheme for some field k, i.e. a morphism of schemes

$$(X, \mathcal{O}_X) \to (\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)})$$

is fixed, show that every residue field $k(x)$ is naturally a field extension $k \subset k(x)$. A point $x \in X$ is *rational* if this extension is bijective, i.e. $k = k(x)$. The *set of rational points* is denoted by $X(k)$. Show that $X(k)$ can be described as the set of k-morphisms $\mathrm{Spec}(k) \to (X, \mathcal{O}_X)$, i.e. morphisms such that the composition

$$(\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)}) \to (X, \mathcal{O}_X) \to (\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)})$$

is the identity of schemes.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 16. *Small non-affine schemes* (3 extra points)
Construct an example of a scheme $(X, \mathcal{O}_X)$ which is not affine and for which X is finite and as small as possible.