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Winter term 2022/23

Exercises, Algebraic Geometry I – Week 4

Exercise 17. *Zariski tangent space* (4 points)
Let $(X, \mathcal{O}_X)$ be an scheme. For a point $x \in X$ the quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$ is considered as a vector space over the residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. The *Zariski tangent space* T_x of X at $x \in X$ is defined as the dual of this vector space, i.e.

$$T_x := (\mathfrak{m}_x/\mathfrak{m}_x^2)^* = \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)).$$

Assume $(X, \mathcal{O}_X)$ is a k-scheme, where k is a field, and denote the *ring of dual numbers* $k[t]/(t^2)$ by $k[\varepsilon]$.

(i) Show that giving a morphism $(\mathrm{Spec}(k[\varepsilon]), \mathcal{O}_{\mathrm{Spec}(k[\varepsilon])}) \to (X, \mathcal{O}_X)$ that is compatible with the morphisms to $(\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)})$ is equivalent to giving a rational point $x \in X$ and an element $v \in T_x$.

(ii) Calculate T_0 for $\mathbb{A}^n_k$ and for $\mathrm{Spec}(k[x, y]/(y^2 + x^3)) \subseteq \mathbb{A}^2_k$.

Exercise 18. *Noetherian topological spaces* (4 points)
A topological space X is called *Noetherian* if it satisfies the descending chain condition for closed subsets, i.e., if for every chain of closed subsets $V_1 \supseteq \ldots \supseteq V_i \supseteq \ldots$, there exists an $r \geq 1$ such that $V_i = V_r$ for all $i \geq r$.

(i) Show that X is Noetherian if and only if every open subset $U \subseteq X$ is quasi-compact. In particular, Noetherian spaces are quasi-compact and quasi-separated.

(ii) Show that any subset (with the subspace topology) of a Noetherian space X is Noetherian.

(iii) An *irreducible component* of a topological space X is an irreducible subset which is maximal with respect to inclusion of irreducible subsets. Show that, in general, irreducible components are closed. Show that a Noetherian space has finitely many irreducible components.

(iv) Show that the underlying topological space of a Noetherian scheme $(X, \mathcal{O}_X)$ is Noetherian. Give an example that shows that the converse is not true in general.

Exercise 19. *A non-affine open subscheme of an affine scheme* (3 points)
Let k be a field and consider the affine plane $\mathbb{A}^2_k = \mathrm{Spec}\ k[x, y]$. Let $0 \in \mathbb{A}^2_k$ be the closed point corresponding to the maximal ideal (x, y) and let $U := \mathbb{A}^2_k \setminus 0$.

(i) Show that restriction of sections induces an isomorphism $H^0(\mathbb{A}^2_k, \mathcal{O}_{\mathbb{A}^2_k}) \xrightarrow{\sim} H^0(U, \mathcal{O}_U)$.

(ii) Deduce that U is not an affine scheme.

Please turn over

Due 04.11.2022, 2pm

Exercise 20. *Glueing schemes and morphisms.* (5 points)
A *glueing datum* is a quadruple $(I, \{X_i\}_{i\in I}, \{U_{ij}\}_{i,j\in I}, \{\varphi_{ij}\}_{i,j\in I})$ where I is an index set, each X_i is a scheme, each $U_{ij} \subseteq X_i$ is an open subscheme, and each $\varphi_{ij} : U_{ij} \to U_{ji}$ is an isomorphism of schemes such that for all $i, j, k \in I$ the following three conditions hold:

(a) $U_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}_{X_i}$.

(b) $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$.

(c) $\varphi_{ik}|_{U_{ij}\cap U_{ik}} = \varphi_{jk}|_{U_{ji}\cap U_{jk}} \circ \varphi_{ij}|_{U_{ij}\cap U_{ik}}$

For $x \in X_i$ and $x' \in X_j$ we write $x \sim x'$ if and only if $x \in U_{ij}$, $x' \in U_{ji}$, and $\varphi_{ij}(x) = x'$.

(i) Show that $\sim$ is an equivalence relation. Define $X := (\coprod X_i)/\sim$ and let $\varphi_i : X_i \to X$ be the natural map.

(ii) Define a subset $U \subseteq X$ to be open if and only if $\varphi_i^{-1}(U)$ is open for all i. Show that this defines a topology on X such that each φ_i is a homeomorphism onto its image. Define $U_i := \varphi_i(X_i)$.

(iii) Define $\mathcal{O}_{U_i} := \varphi_{i,*}\mathcal{O}_{X_i}$. Use the φ_{ij} to glue the $\mathcal{O}_{U_i}$ to a sheaf of rings $\mathcal{O}_X$ on X such that $(X, \mathcal{O}_X)$ is a scheme.

(Hint: Recall Exercise 3.)

(iv) Show that the scheme X satisfies the following universal property: For every scheme Y and every collection of morphisms of schemes $f_i : X_i \to Y$ such that $f_j|_{U_{ji}} \circ \varphi_{ij} = f_i|_{U_{ij}}$ for all $i, j \in I$, there exists a unique morphism of schemes $f : X \to Y$ such that $f_i = f \circ \varphi_i$ for all $i \in I$.

(Remark: As a special case, note that if X' is a scheme, $X' = \cup_{i\in I} X_i$ is an open cover with $U_{ij} := X_i \cap X_j$ and $\varphi_{ij} := \mathrm{id}_{U_{ij}}$, then the inclusions $X_i \to X'$ define an isomorphism $f : X \to X'$ by Exercise 14.)

Exercise 21. *Reduced schemes and reduction of schemes* (4 points)
Let X be a scheme. The *reduction* of X is a reduced scheme X_{red} together with a morphism $\iota : X_{\text{red}} \to X$ such that every morphism $Z \to X$ from a reduced scheme Z factors uniquely through ι.

(i) Show that if $X = \mathrm{Spec}(A)$ is affine, then $\mathrm{Spec}(A/\mathfrak{N}(A))$, where $\mathfrak{N}(A)$ is the nilradical of A, together with the morphism ι induced by the quotient map $A \to A/\mathfrak{N}(A)$ is the reduction of X.

(ii) Show that every scheme admits a (necessarily unique) reduction. Show that ι is a homeomorphism of topological spaces.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 22. *Right-adjoint to global sections* (3 extra points)
In the lecture, we have seen that $\mathrm{Spec}(-)$ is right-adjoint to the functor that maps a locally ringed space to its ring of global sections. Can you find the right-adjoint to the functor that maps a ringed space to its ring of global sections?