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Winter term 2022/23

Exercises, Algebraic Geometry I – Week 5

Exercise 23. *Factoring morphisms through subschemes* (4 points)
Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes.

(i) Let $(\iota, \iota^\sharp) : (U, \mathcal{O}_U) \to (Y, \mathcal{O}_Y)$ be the open immersion of an open subscheme. Show that $(f, f^\sharp)$ factors through $(\iota, \iota^\sharp)$ if and only if $f(X) \subseteq U$ (as sets).

(ii) Let $(\iota, \iota^\sharp) : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ be the closed immersion of a closed subscheme. Show that $(f, f^\sharp)$ factors through $(\iota, \iota^\sharp)$ if and only if $f^\sharp$ factors through $\iota^\sharp$. Show that this is automatically satisfied if X is reduced and $f(X) \subseteq Z$ (as sets).

(Remark: The smallest closed subscheme of Y through which $(f, f^\sharp)$ factors is also called *scheme-theoretic image* of $(f, f^\sharp)$)

(iii) (+1 extra point) Give an example of a morphism of schemes such that the underlying set of its scheme-theoretic image is not the closure of its set-theoretic image.

Exercise 24. *Generic points and dominant rational maps* (3 points)
A point $\eta \in X$ of a topological space X is called *generic* if $\overline{\eta} = X$ and a continuous map $f : X \to Y$ is called *dominant* if $f(X)$ is dense in Y.

(i) Show that every integral scheme X admits a unique generic point η_X. Show that the local ring $\mathcal{O}_{X,\eta_X}$ is a field. This field is called *function field* of X and is denoted by $k(X)$.

(ii) Show that a morphism $f : X \to Y$ between integral schemes is dominant if and only if $f(\eta_X) = \eta_Y$. Deduce that a dominant morphism of integral schemes induces a field extension $k(Y) \hookrightarrow k(X)$.

Exercise 25. *Immersions and base change* (4 points)
Let $f : X \to Y$ be a morphism of schemes.

(i) Show that f is an open (resp. closed) immersion if and only if there exists an open affine cover $Y = \cup_{i \in I} U_i$ such that each $f^{-1}(U_i) \to U_i$ is an open (resp. closed) immersion.

(ii) Show that f is an open (resp. closed) immersion if and only if for all morphisms of schemes $g : Z \to Y$, the induced morphism $Z \times_Y X \to Z$ is an open (resp. closed) immersion.

Please turn over

Due 11.11.2022, 2pm

Exercise 26. *Integral and irreducible fibres* (4 points)
Find examples for the following phenomena:

(i) Show that there exist surjective morphisms $X \to Y$ with Y integral and such that all fibres X_y are irreducible without X being irreducible.

(ii) Show that for every algebraically closed field k, there exist morphisms $X \to \mathrm{Spec}(k[x])$ with X integral, the generic fibre X_η non-empty and integral, but no closed fibre integral.

(iii) Show that there exist morphisms $X \to \mathrm{Spec}(\mathbb{Q}[x])$ with X integral and infinitely many irreducible and infinitely many reducible closed fibres. What happens for the geometric closed fibres in your example?

(iv) Show that there exist morphisms $f : X \to Y$ with X and Y integral whose geometric generic fiber is not reduced.

Exercise 27. *Normalization* (5 points)
A scheme X is *normal* if all its local rings $\mathcal{O}_{X,x}$ are integrally closed domains. The *normalization* of an integral scheme X is an irreducible normal scheme $\tilde{X}$ together with a dominant morphism $\nu : \tilde{X} \to X$ such that every dominant morphism $Z \to X$ from an irreducible normal scheme Z factors uniquely through ν.

(i) Let $X = \mathrm{Spec}(A)$ for an integral domain A, let $\tilde{X} = \mathrm{Spec}(\tilde{A})$, where $\tilde{A}$ is the integral closure of A in its field of fractions $K(X)$, and let $\nu : \tilde{X} \to X$ be the morphism induced by the inclusion $A \subset \tilde{A}$. Show that $\tilde{X}$ together with ν is the normalization of X.

(ii) Show that every integral scheme admits a (necessarily unique) normalization.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 28. *Monomorphisms of schemes* (+ 3 extra points)
Recall that a *monomorphism* in a category $\mathcal{C}$ is a morphism $f : X \to Y$ such that for any two morphisms $g_1, g_2 : Y \to Z$, the equality $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

(i) Show that $f : X \to Y$ is a monomorphism in $\mathcal{C}$, if and only if the fiber product $X \times_Y X$ exists and the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism.

(ii) Let $(f, f^\sharp) : X \to Y$ be a morphism of schemes. Assume that f is injective and for all points $x \in X$, the morphism $f_x^\sharp$ is surjective. Show that $(f, f^\sharp)$ is a monomorphism in (Sch). Deduce that open and closed immersions of schemes are monomorphisms.

(iii) Find a monomorphism of schemes which is not a composition of open and closed immersions.