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Winter term 2022/23

Exercises, Algebraic Geometry I – Week 6

Exercise 29. *Properties of diagonals* (4 points)
Let $\mathcal{P}$ be a property of morphisms of schemes that is satisfied by isomorphisms. We say that a morphism $f : X \to Y$ satisfies $\Delta_{\mathcal{P}}$ if its diagonal satisfies $\mathcal{P}$. Show the following:

(i) If $\mathcal{P}$ satisfies (BC), then $\Delta_{\mathcal{P}}$ satisfies (BC).

(ii) If $\mathcal{P}$ satisfies (BC) and (COMP), then $\Delta_{\mathcal{P}}$ satisfies (COMP).

(iii) If $\mathcal{P}$ satisfies (LOCT), then $\Delta_{\mathcal{P}}$ satisfies (LOCT).

(iv) If $\mathcal{P}$ satisfies (BC) and (COMP), $f : X \to Y, g : X \to Z$ are morphisms of schemes over S that satisfy $\mathcal{P}$ and $X \to S$ satisfies $\Delta_{\mathcal{P}}$, then $(f, g) : X \to Y \times_S Z$ satisfies $\mathcal{P}$.

Exercise 30. *Topological properties of morphisms* (4 points)
Show the following:

(i) "surjective" satisfies (BC).

(ii) "injective","bijective" do not satisfy (BC).

(iii) (+ 1 extra point) "finite fibers" does not satisfy (BC).

(iv) "closed" does not satisfy (BC).

(v) "quasi-compact" and "quasi-separated" do not satisfy (LOCS).

Exercise 31. *Morphisms locally of finite type* (5 points)
Consider the property $\mathcal{P} =$ "locally of finite type":

(i) Show that "f_V is locally of finite type" is an affine-local property of open affine subschemes $V \subseteq Y$. Observe that this implies that $\mathcal{P}$ satisfies (LOCT).

(ii) Show that $\mathcal{P}$ satisfies (COMP), (BC), and (CANC).

(iii) Show that $\mathcal{P}$ satisfies (LOCS).

Exercise 32. *Diagonals are immersions* (4 points)
Let $f : X \to Y$ be a morphism of schemes.

(i) Show that if f is affine, then $\Delta_{X/Y}$ is a closed immersion. In other words, affine morphisms are separated.

(ii) Show that $\Delta_{X/Y}$ is an immersion. Deduce that $\Delta_{X/Y}$ is a closed immersion if and only if its set-theoretic image is closed.

(iii) Show that if f is an immersion, then $\Delta_{X/Y}$ is an isomorphism. [1]

Please turn over

Due 18.11.2022, 2pm

[1] This occurred as a special case of the bonus exercise on the previous sheet, but you can prove it directly.

(iv) For each of the following properties $\mathcal{P}$, show that if $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ satisfies $\mathcal{P}$ and g is separated, then f satisfies $\mathcal{P}$:

"quasi-compact", "quasi-separated", "closed immersion", "of finite type", "quasi-finite", "affine", "finite", "separated", "proper"

Exercise 33. *Morphisms into separated schemes* (4 points)
Let $f_1, f_2 : X \to Y$ be morphisms of schemes over S. In this exercise, we want to study the locus of points in X where f_1 and f_2 coincide and endow it with a scheme structure.

(i) Let $\iota : \mathrm{eq}(f_1, f_2) \to X$ be the equalizer of f_1 and f_2, i.e., $\mathrm{eq}(f_1, f_2)$ is a scheme and ι is a morphism with $f_1 \circ \iota = f_2 \circ \iota$ satisfying the following universal property:

For all $g : Z \to X$ such that $f_1 \circ g = f_2 \circ g$, the morphism g factors uniquely through ι.

Show that ι exists and that it is an immersion.

(Hint: Rephrase the universal property of ι as a universal property of a fiber product)

(ii) Assume that Y is separated over S. Show that ι is a closed immersion.

(iii) Conclude that if X is reduced, Y is separated, and f_1 and f_2 coincide on an open dense subset of X, then $f_1 = f_2$.

(iv) Give a counterexample to (iii) if Y is not separated.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 34. *Base change as a functor* (+ 3 extra points)
Consider a morphism of schemes $S \to T$. Every S-scheme X, i.e. every morphism $X \to S$, yields via composition with $S \to T$ a T-scheme $X \to S \to T$ which we shall denote ${}_TX$. Conversely, to every T-scheme $Y \to T$ base change defines an S-scheme $Y_S := S \times_T Y \to S$.

(i) Show that this defines two functors

$$_T(\ )\colon (Sch/S) \to (Sch/T),\ X \mapsto {}_TX \text{ and } (\ )_S\colon (Sch/T) \to (Sch/S),\ Y \mapsto Y_S$$

which are adjoint to each other. More precisely, ${}_T(\ )$ is left adjoint to $(\ )_S$, i.e. ${}_T(\ ) \dashv (\ )_S$.

For any S-scheme X consider the functor

$$\begin{aligned} \mathbf{Res}_{S/T}(X)\colon \quad (Sch/T)^{\mathrm{op}} &\to (Sets) \\ Y &\mapsto \mathrm{Mor}_S(Y_S, X). \end{aligned}$$

If this functor is representable by a T-scheme, which will be denoted by $\mathrm{Res}_{S/T}(X)$, it is called the *Weil restriction* and satisfies $h_{\mathrm{Res}_{S/T}(X)} \cong \mathbf{Res}_{S/T}(X)$.

One can show that the Weil restriction exists for finite field extensions $S = \mathrm{Spec}(K) \to T = \mathrm{Spec}(k)$ and $X = \mathrm{Spec}(A)$, where A is a finite type K-algebra. In this case, one writes $\mathrm{Res}_{K/k}(X)$ for the Weil restriction.

(ii) Set $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}^1_{\mathbb{C}} \setminus 0)$. Show that the rational points of $\mathbb{S}$ can be described as

$$\mathbb{S}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R},\ a^2 + b^2 \neq 0 \right\}.$$

(Remark: Usually, i.e. for general $S \to T$ and X, the Weil restriction does not exist.)

Please turn over

Recall:

Let $\mathcal{P}$ be a property of morphisms of schemes. Then, we say that $\mathcal{P}$ satisfies

(COMP) if it is *stable under composition*, that is, if for all morphisms $f : X \to Y$ and $g : Y \to Z$ that satisfy $\mathcal{P}$, also $g \circ f$ satisfies $\mathcal{P}$.

(CANC) if it is *cancellable*, that is, if for all morphisms $f : X \to Y$ and $g : Y \to Z$ such that $g \circ f$ satisfies $\mathcal{P}$, also f satisfies $\mathcal{P}$.

(BC) if it is *stable under base change*, that is, if for all morphisms $f : X \to Y$ and $g : Y' \to Y$ such that f satisfies $\mathcal{P}$, also $f_{Y'}$ satisfies $\mathcal{P}$.

(LOCT) if it is (Zariski-)*local on the target*, that is, if for all morphisms $f : X \to Y$ and all open covers $Y = \cup_{i \in I} V_i$, the morphism f satisfies $\mathcal{P}$ if and only if f_{V_i} satisfies $\mathcal{P}$ for all $i \in I$.

(LOCS) if it is (Zariski-)*local on the source*, that is, if for all morphisms $f : X \to Y$ and all open covers $X = \cup_{i \in I} U_i$, the morphism f satisfies $\mathcal{P}$ if and only if $f|_{U_i}$ satisfies $\mathcal{P}$ for all $i \in I$.

Note that $\Delta_{\mathcal{P}}$ is also a property of a morphism $f\colon X \to Y$. For example, $\Delta_{\mathcal{P}}$ satisfies (BC) means, given any base change $f_{Y'}\colon X_{Y'} \to Y'$ of f, $\Delta_{X_{Y'}/Y'}$ satisfies $\mathcal{P}$. Similarly $\Delta_{\mathcal{P}}$ satisfies (LOCT) means that for every open covering $Y = \cup_{i \in I} V_i$, the map $\Delta_{X/Y}$ satisfies $\mathcal{P}$ if and only if the map $\Delta_{X_{V_i}/V_i}$ satisfies $\mathcal{P}$ for all $i \in I$ and so on.