Dr. Gebhard Martin
Dr. Yajnaseni Dutta

Winter term 2022/23

Exercises, Algebraic Geometry I – Week 7

Exercise 35. *Valuation rings* (5 points)
In this exercise, we want to understand why valuation rings are called "valuation"rings and why discrete valuation rings (DVRs) are indeed "discrete".

(i) Let A be a subring of a field K. Assume that for every $x \in K^\times$, either $x \in A$ or $x^{-1} \in A$. Show that A is a valuation ring with field of fractions K. [Bonus Exercise (+2 extra points): Show that the converse is true as well, i.e., if A is a valuation ring with field of fractions K, then for every $x \in K^\times$ either $x \in A$ or $x^{-1} \in A$.]

(ii) Let A be a valuation ring in a field K. Let $\Gamma = K^\times/A^\times$ be the group (called the *value group)* formed by quotienting the multiplicative group $K^\times$ by the group $A^\times$ of invertible elements of A. Let $v : K^\times \to \Gamma$ be the quotient map.

Recall that a totally ordered abelian group is an abelian group G with a total order $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ for any $a, b, c \in G$. Now for $\gamma, \gamma' \in \Gamma$, represented by $x, x' \in K^\times$, define $\gamma \leq \gamma'$ if $\frac{x'}{x} \in A$. Show that $\leq$ is well-defined and turns Γ into a totally ordered abelian group. Show that v is a *valuation* (i.e. a group homomorphism satisfying $v(a + b) \geq \min\{v(a), v(b)\}$)

Show that $\leq$ is a well-defined total order on Γ. Show that v is a *valuation* (i.e. a group homomorphism satisfying $v(a + b) \geq \min\{v(a), v(b)\}$).

[Hint: Use the new definition of valuation ring as in (i).]

(iii) Conversely, let Γ be a totally ordered abelian group and let $v : K^\times \to \Gamma$ be a valuation. Show that $A = \{a \in K^\times \mid v(a) \geq 0\} \cup \{0\}$ is a valuation ring in K with maximal ideal $\mathfrak{m}_A := \{a \in K^\times \mid v(a) > 0\} \cup \{0\}$.

(iv) Show that in a valuation ring every finitely generated ideal is principal. Thus a Noetherian valuation ring is a local principal ideal domain.

(v) Show that a valuation ring A is Noetherian if and only if the associated valuation factors through a cyclic subgroup.

[Hint: Krull's Intersection Theorem]

(Note: In particular, either $v(a) = 0$ for all $a \in A$ and then $A = K$, or $v(a) > 0$ for some $a \in A$ and then v factors through a group isomorphic to $\mathbb{Z}$, hence the name "discrete" valuation ring.)

Exercise 36. *Morphisms from regular curves to proper schemes* (4 points)
The goal of this exercise is to get a taste of how the valuative criterion for properness is used in practice.

(i) Let S be a locally Noetherian scheme, let X and Y be schemes over S, assume Y is locally of finite type over S, and let $x \in X$ be a point. Show that every morphism of schemes $\operatorname{Spec} \mathcal{O}_{X,x} \to Y$ over S is induced by a morphism of schemes $U \to Y$ over S, where $U \subseteq X$ is an open affine neighbourhood of x in X.

Due 25.11.2022, 2pm

(ii) In addition to the assumptions in (i), assume that $\mathcal{O}_{X,x}$ is a valuation ring, X is integral and Y is proper over S. Show that for every open subscheme $U \subseteq X$ such that x is contained in the closure of U, any morphism of S-schemes $U \to Y$ extends uniquely to a morphism $f' \colon U' \to Y$ over S where $U \subset U' \subseteq X$ is a bigger open set containing x.

Exercise 37. *Adjoint functors* $f^* \dashv f_*$ (4 points)
Consider a morphism of ringed spaces $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. Show that f^* is left adjoint to f_*, i.e. for all $\mathcal{F} \in \mathrm{Mod}(X, \mathcal{O}_X)$ and $\mathcal{G} \in \mathrm{Mod}(Y, \mathcal{O}_Y)$ there exists an isomorphism (functorial in $\mathcal{F}$ and $\mathcal{G}$):

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

Exercise 38. $M \mapsto \tilde{M}$ *and adjunction* (4 points)
Let X be the affine scheme $\mathrm{Spec}(A)$ and consider an A-module M and a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. Show that $(A\text{-mod}) \to \mathrm{Mod}(X, \mathcal{O}_X)$, $M \mapsto \tilde{M}$ is left adjoint to $\mathrm{Mod}(X, \mathcal{O}_X) \to (A\text{-mod})$, $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$, i.e. that there exists a functorial (in M and $\mathcal{F}$) isomorphism

$$\mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}).$$

Exercise 39. *Projection formula* (3 points)
Consider a morphism of ringed spaces $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $\mathcal{F} \in \mathrm{Mod}(X, \mathcal{O}_X)$ and $\mathcal{G} \in \mathrm{Mod}(Y, \mathcal{O}_Y)$. Suppose $\mathcal{G}$ is locally free of finite rank. Show that there exists a natural isomorphism

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}.$$

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 40. *Valuative criteria with DVRs* (+ 6 extra points)
The goal of this exercise is to prove a valuative criterion for morphisms of finite type between locally Noetherian schemes.

(i) Let A be a Noetherian local integral domain with field of fractions K. Let L be a finitely generated field extension of K. Show that there exists a discrete valuation ring B with field of fractions L that dominates A. You can follow the following steps:

  (a) If L is not finite over K, let $x_1, \dots, x_n$ be a transcendence basis of L over K and replace A by a suitable localization of $A[x_1, \dots, x_n]$ to reduce to the case where L is finite over K.

  (b) Take any valuation ring dominating A in K and let v be the corresponding valuation. Let $x_1, \dots, x_n$ be a minimal set of generators of the maximal ideal $\mathfrak{m}$ of A and order the x_i such that $v(x_n) = \min\{v(x_1), \dots, v(x_n)\}$. Set $A' = A[\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}] \subseteq K$. Show that $\mathfrak{m}A' \neq A'$. Show that the localization of A' at a minimal prime over $\mathfrak{m}A'$ has dimension 1 and dominates A.

  (c) Finish the proof by applying the Krull–Akizuki theorem. (see for example TAG 00PG on the Stacks project)

(ii) Let $f : X \to Y$ be a morphism of finite type and assume that Y is locally Noetherian.

  (a) Show that f is universally closed if and only if f satisfies the existence part of the valuative criterion for DVRs.

  (b) Show that f is separated if and only if f satisfies the uniqueness part of the valuative criterion for DVRs.

  (c) Show that f is proper if and only if f satisfies existence and uniqueness of the valuative criterion for DVRs.