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Winter term 2022/23

Exercises, Algebraic Geometry I – Week 8

Exercise 41. *Support* (4 points)
Recall the notions of support of a section of a sheaf and subsheaf with supports from Exercise 8. If $\mathcal{F}$ is a sheaf of Abelian groups on a topological space X, then

$$\operatorname{supp}(\mathcal{F}) \coloneqq \{x \in U \mid \mathcal{F}_x \neq 0\}.$$

(i) Let A be a ring, let M be an A-module, let $X = \operatorname{Spec} A$, and let $\mathcal{F} = \widetilde{M}$. For any $m \in M$, show that $\operatorname{supp}(m) = V(\operatorname{Ann} m)$, where $\operatorname{Ann} m = \{a \in A \mid am = 0\}$ is the annihilator of m.

(ii) Assume that A is Noetherian and M is finitely generated. Show that $\operatorname{supp}(\mathcal{F}) = V(\operatorname{Ann} M)$, where $\operatorname{Ann} M = \{a \in A \mid aM = 0\}$.

(iii) Deduce that the support of a coherent sheaf on a Noetherian scheme is closed.

(iv) For any ideal $I \subseteq A$, set $\Gamma_I(M) \subseteq M$ via $\Gamma_I(M) \coloneqq \{m \in M \mid I^n m = 0 \text{ for some } n > 0\}$. Assume that A is Noetherian and M is an arbitrary A-module. Show that $\widetilde{\Gamma_I(M)} \cong \mathcal{H}^0_Z(\mathcal{F})$, where $Z = V(I)$ and $\mathcal{F} = \widetilde{M}$.

(v) Deduce that if X is a Noetherian scheme, $Z \subseteq X$ is a closed subset and $\mathcal{F}$ is a (quasi-)coherent sheaf on X, then $\mathcal{H}^0_Z(\mathcal{F})$ is also (quasi-)coherent.

Exercise 42. *Fiber dimension* (4 points)
Let X be a Noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. We will consider the function

$$\varphi(x) \coloneqq \dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x),$$

where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field of the point $x \in X$. Use Nakayama lemma to prove the following statements.

(i) The function φ is upper semi-continuous, i.e. for any $n \in \mathbb{Z}$ the set $\{x \in X \mid \varphi(x) \geq n\}$ is closed.

(ii) If $\mathcal{F}$ is locally free and X is connected, then φ is a constant function.

(iii) Conversely, if X is reduced and φ is constant, then $\mathcal{F}$ is locally free.

Exercise 43. *f^* and $\otimes$ are only right exact* (4 points)
Let $(X, \mathcal{O}_X)$ be a ringed space and consider $\mathcal{F}, \mathcal{G} \in \operatorname{Mod}(X, \mathcal{O}_X)$. Show that for all $x \in X$ there exists a natural isomorphism

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x.$$

Prove that $\mathcal{F} \otimes_{\mathcal{O}_X} (\ ) \colon \operatorname{Mod}(X, \mathcal{O}_X) \to \operatorname{Mod}(X, \mathcal{O}_X)$ and $f^* \colon \operatorname{Mod}(Y, \mathcal{O}_Y) \to \operatorname{Mod}(X, \mathcal{O}_X)$ for a morphism of ringed spaces $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ are both right exact functors. Describe examples showing that in general they are not left exact.

Due 02.12.2022, 2pm

Exercise 44. *(Non-)Functoriality of* Proj (4 points)
Let $A = \bigoplus_{d=0}^{\infty} A_d$ and $B = \bigoplus_{d=0}^{\infty} B_d$ be graded rings. Recall that a morphism of rings $\varphi : A \to B$ is called *graded* if $\varphi(A_d) \subseteq B_d$.

(i) Let $\varphi : A \to B$ be a graded morphism of rings. Let $U = \{\mathfrak{p} \in \text{Proj } B \mid \varphi(A_+) \not\subseteq \mathfrak{p}\}$. Show that $U \subseteq \text{Proj } B$ is an open subset and show that φ determines a morphism of schemes $U \to \text{Proj } A$.

(ii) Assume that $A_d = B_d$ and let $\varphi : A \to B$ be a graded morphism of rings which induces the identity for $d \gg 0$. Show that $U = \text{Proj } B$ and the induced map $U \to \text{Proj } A$ is an isomorphism.

Parts (ii) and (iii) of the following exercise use twisting sheaves and the description of rational points of projective space, both of which will be treated in the lecture on Monday.

Exercise 45. *Products of* Proj *and Segre embedding* (4 points)
Let $B = \bigoplus_{d=0}^{\infty} B_d$ and $C = \bigoplus_{d=0}^{\infty} C_d$ be two graded rings with $A := B_0 \cong C_0$. Consider $B \times_A C := \bigoplus_{d=0}^{\infty} B_d \otimes_A C_d$ and the schemes $X := \text{Proj}(B)$ and $Y := \text{Proj}(C)$.

(i) Show that $X \times_{\text{Spec}(A)} Y \cong \text{Proj}(B \times_A C)$.

(ii) Prove that under this isomorphism $\mathcal{O}(1)$ on $\text{Proj}(B \times_A C)$ is isomorphic to $p_1^*\mathcal{O}_X(1) \otimes p_2^*\mathcal{O}_Y(1)$, where p_1 and p_2 are the two projections from $X \times_{\text{Spec}(A)} Y$.

(iii) Now, specialize to $X = \mathbb{P}^{n_1}_{A_0} = \text{Proj } A_0[x_0, \ldots, x_{n_1}]$ and $Y = \mathbb{P}^{n_2}_{A_0} = \text{Proj } A_0[y_0, \ldots, y_{n_2}]$. Show that the surjective graded morphism

$$A_0[z_{00}, \ldots, z_{n_1 n_2}] \to A_0[x_0, \ldots, x_{n_1}] \times_{A_0} A_0[y_0, \ldots, y_{n_2}]$$

that maps z_{ij} to $x_i \otimes y_j$ induces a closed immersion $X \times_{\text{Spec } A} Y \to \mathbb{P}^N_A$. This closed immersion is called *Segre embedding.* What is N? Describe the Segre embedding on rational points if A_0 is a field.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 46. *Multiplicative group actions and gradings* (+ 4 extra points)
We define the *multiplicative group scheme* $\mathbb{G}_m$ as the scheme $\text{Spec } \mathbb{Z}[x, x^{-1}]$ together with the three morphisms

$$\begin{array}{rcl} \text{“neutral element” } e : & \text{Spec } \mathbb{Z} & \to \mathbb{G}_m \\ & 1 & \leftarrow\!\shortmid x \\ \text{“multiplication” } m : & \mathbb{G}_m \times_{\text{Spec } \mathbb{Z}} \mathbb{G}_m & \to \mathbb{G}_m \\ & x \otimes x & \leftarrow\!\shortmid x \\ \text{“inverse” } i : & \mathbb{G}_m & \to \mathbb{G}_m \\ & x^{-1} & \leftarrow\!\shortmid x \end{array}$$

(i) Rephrase the definition of a group action on a set in terms of commutative diagrams. Use these diagrams to define $\mathbb{G}_m$-actions on schemes.

(ii) Show that giving a $\mathbb{G}_m$-action on an affine scheme Spec A is the same as giving a $\mathbb{Z}$-grading $A = \bigoplus_{d \in \mathbb{Z}} A_d$.

(iii) Can you find necessary and sufficient conditions on the $\mathbb{G}_m$-action that guarantee $A_d = 0$ for $d < 0$ (i.e. such that A is a graded ring in the sense of the lecture)? Try to find the $\mathbb{G}_m$-action corresponding to the grading on $A_0[x_0, \ldots, x_n]$ that yields projective n-space.