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Winter term 2022/23

Exercises, Algebraic Geometry I – Week 9

Exercise 47. *Qcqs lemma* (4 points)
Let X be a scheme, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. For $s \in \Gamma(X, \mathcal{L})$, define $X_s \coloneqq \{x \in X \mid s_x \notin \mathfrak{m}_x \mathcal{L}_x\}$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$.

(i) Show that $X_s \subseteq X$ is open.

(ii) Assume that X is quasi-compact and let $t \in \Gamma(X, \mathcal{F})$ such that $t|_{X_s} = 0$. Show that there exists an integer $n > 0$ such that $t \otimes s^{\otimes n} = 0 \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$.

(iii) Assume that X is quasi-compact and quasi-separated. Show that for every section $t' \in \Gamma(X_s, \mathcal{F})$, there exists $n > 0$ and a section $t \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ such that $t|_{X_s} = t' \otimes s|_{X_s}^{\otimes n}$

(Hint: Reduce to the case where X is an affine scheme and $\mathcal{L}$ is the structure sheaf.)

Exercise 48. *Veronese embedding* (4 points)
Let $A = \bigoplus_{d \geq 0} A_d$ be a graded ring and $n > 0$. Let $A^{(n)}$ denote the graded ring defined by $A^{(n)} \coloneqq \bigoplus_{d \geq 0} A_d^{(n)}$ with $A_d^{(n)} \coloneqq A_{dn}$.

(i) Show that there exists an isomorphism $\varphi \colon \operatorname{Proj}(A) \xrightarrow{\sim} \operatorname{Proj}(A^{(n)})$ with $\varphi^* \mathcal{O}(1) \cong \mathcal{O}(n)$.

(ii) Consider the case $A = A_0[x_0, \ldots, x_r]$. Show that the surjection $A_0[y_0, \ldots, y_N] \twoheadrightarrow A^{(n)}$ mapping y_i to the i-th monomial of degree n in the variables x_i defines a closed embedding
$$v_n : \mathbb{P}^r_{A_0} \cong \operatorname{Proj}(A) \cong \operatorname{Proj}(A^{(n)}) \hookrightarrow \mathbb{P}^N_{A_0}.$$
Find N. Show that for k a field, on k-rational points the morphism v_n is given by $[\lambda_0 : \cdots : \lambda_r] \mapsto [\lambda_0^n : \cdots : \lambda^I : \cdots : \lambda_r^n]$ with λ^I running through all monomials of degree n.

Exercise 49. *Weighted projective space* (4 points)
Let $n, a_0, \ldots, a_n \geq 1$ be integers and A_0 a ring. Let $A = A_0[x_0, \ldots, x_n]$. Equip A with the grading such that x_i is homogeneous of degree a_i. We define $\mathbb{P}_{A_0}(a_0, \ldots, a_n) \coloneqq \operatorname{Proj} A_0[x_0, \ldots, x_n]$ and call it *weighted projective space with weights* $(a_0, \ldots, a_n)$ over A_0.

(i) Let $A_0[y_0, \ldots, y_n]$ be the polynomial ring with the standard grading. Show that the graded A_0-algebra morphism
$$A_0[x_0, \ldots, x_n] \to A_0[y_0, \ldots, y_n]; \qquad x_i \mapsto y_i^{a_i}$$
induces a finite surjective morphism $\pi : \mathbb{P}^n_{A_0} \to \mathbb{P}^n(a_0, \ldots, a_n)$.

(ii) Show that $\mathbb{P}_{A_0}(a_0, a_1) \cong \mathbb{P}^1_{A_0}$ for all $a_0, a_1 \geq 1$ (not necessarily via π).

Due 09.12.2022, 2pm

(iii) Show that $\mathbb{P}_{A_0}(1,1,2) \cong V_+(y_1^2 - y_0 y_2) \subseteq \mathbb{P}^3_{A_0}$. Deduce that $\mathbb{P}_{A_0}(1,1,2)$ is not isomorphic to $\mathbb{P}^2_{A_0}$ over Spec A_0.

(Hint: Reduce to the case where A_0 is a field k and calculate the Zariski tangent space of $V_+(y_1^2 - y_0 y_2)$ at the k-rational point $[0:0:0:1]$)

Exercise 50. *Projection from a point* (4 points)

(i) Let k be a field. Show that the (graded) inclusion $k[x_0, \dots, x_{n-1}] \hookrightarrow k[x_0, \dots, x_n]$ defines a morphism $\pi : \mathbb{P}^n_k \setminus \{[0 : \dots : 0 : 1]\} \to \text{Proj } k[x_0, \dots, x_{n-1}] = \mathbb{P}^{n-1}_k$. This is called the *projection from the point* $[0 : \dots : 0 : 1]$ (to the hyperplane $V_+(x_n)$).

(ii) Describe the k-rational points of the fibers of π over k-rational points of $\mathbb{P}^{n-1}_k$.

(iii) Show that for any homogeneous polynomial $f \in k[x_0, \dots, x_n]$ of degree 1 and a point $P \in D_+(f)$, there exists an automorphism $\sigma : \mathbb{P}^n_k \to \mathbb{P}^n_k$ such that $\sigma(V_+(f)) = V_+(x_n)$ and $\sigma(P) = [0 : \dots : 0 : 1]$.
(Hint: first try an explicit point P and a hyperplane $V_+(f)$ of your choice.)

(iv) Let $\mathbb{P}^1_k = \text{Proj } k[x_0, x_1]$, $\mathbb{P}^3_k = \text{Proj } k[y_0, y_1, y_2, y_3]$, and $\mathbb{P}^2_k = \text{Proj } k[z_0, z_1, z_2]$. Find a homogeneous ideal $I \subseteq k[z_0, z_1, z_2]$ such that $V_+(I)$ is the image of $\pi \circ v_3$, where $v_3 \colon \mathbb{P}^1 \to \mathbb{P}^3$ is the Veronese map of Exercise 48 and π is the projection from $[0 : 0 : 1 : 0]$.

Exercise 51. *Closed subschemes of products of projective spaces* (4 points)
Let A_0 be a ring, let $n_1, n_2 \geq 1$ be integers and consider

$$X = \mathbb{P}^{n_1}_{A_0} \times_{\text{Spec } A_0} \mathbb{P}^{n_2}_{A_0} = \text{Proj } (A_0[x_0, \dots, x_{n_1}] \times_{A_0} A_0[y, \dots, y_{n_2}]).$$

Let $f_1, \dots, f_m$ be a set of bihomogeneous polynomials, that is, polynomials that are simultaneously homogeneous in the x_i and the y_i. The *closed subscheme Z of X determined by the f_i* is defined as follows: Choose integers m_{xij} and m_{yij} such that all the $x_i^{m_{xij}} f_j$ and $y_i^{m_{yij}} f_j$ have the same (standard) degree, let $I \subseteq A_0[x_0, \dots, x_{n_1}] \times_{A_0} A_0[y, \dots, y_{n_2}]$ be the ideal generated by these new polynomials, and let $Z = V_+(I) \subseteq X$.

(i) Show that the closed subscheme Z does not depend on the chosen m_{xij} and m_{yij}.

(ii) Let $A_0 = k$ be a field and $Z \subseteq \mathbb{P}^1_k \times_{\text{Spec } k} \mathbb{P}^1_k$ a closed subscheme given by the bihomogenous polynomial $x_0 y_1^2 - x_1 y_0^2$. Determine an ideal of image of Z under the Segre embedding $\mathbb{P}^1_k \times_{\text{Spec } k} \mathbb{P}^1_k \to \mathbb{P}^3_k$ of Exercise 45.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 52. *Grothendieck's classification of Locally free sheaves on* $\mathbb{P}^1$ (+ 5 extra points)
Let k be a field. Let $\mathcal{F}$ be a locally free sheaf of rank r on $\mathbb{P}^1_k$.

(i) Show that $\mathcal{F}|_{D_+(x_i)} \cong \mathcal{O}^{\oplus r}_{\mathbb{A}^1_k}$.

(ii) Prove the following normal form for matrices over $k[t, t^{-1}]$: Let M be an $r \times r$ matrix over $k[t, t^{-1}]$ with determinant t^n for some $n \in \mathbb{Z}$. Then, there exist matrices $A \in \text{GL}_r(k[t^{-1}])$ and $B \in \text{GL}_r(k[t])$ such that

$$A \cdot M \cdot B = \text{diag}(t^{a_1}, \cdots, t^{a_r}); \quad \text{the diagonal matrix}$$

with $a_1 \geq a_2 \geq \ldots \geq a_r$, $a_i \in \mathbb{Z}$, and the a_i are uniquely determined by M.
(Hint: Assume that M has polynomial entries and argue by induction on r.)

(iii) Conclude that $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_r)$, where the a_i are integers which are uniquely determined by $\mathcal{F}$ up to reordering.